

A TREATISE
ON
ELEMENTARY TRIGONOMETRY

With Numerous Examples,

AND
QUESTIONS FOR EXAMINATION.

BY

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PREFACE.

THE present Work contains all the Propositions of Plane Trigonometry that do not require the use of De Moivre's Theorem. As it is intended for Junior Students, the Demonstrations are simple and elementary. All necessary explanations are given very fully, but unessential details are purposely omitted. Numerous Examples are appended to all the Chapters. Those in the commencement, being intended for illustration, are very easy, but towards the end they get gradually difficult. The Author trusts it will be found that his Manual removes from the Student many of the difficulties usually experienced in the commencement of Trigonometry. This Work will be fol-

lowed by an advanced Treatise, which is nearly ready, containing all the higher parts of the subject, including Hyperbolic Functions and Spherical Trigonometry.

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ERRATA.

- Page 42, line 15, for *C* read *c*.
 ,, 43, Exercises 1-5, for *C* read *c*.
 ,, 45, line 7, for Log *A* read Log tan *A*.

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A TREATISE ON ELEMENTARY TRIGONOMETRY.

CHAPTER I.

DEFINITIONS, ETC.

1. **The Numerical Measure** of any quantity, such as an angle, a line, &c., is the ratio it bears to a certain standard quantity of the same kind as itself, called the **Unit**. Thus, the **numerical** value of an angle is its ratio to the angular unit. The **numerical** value of a line is the number of linear units (such as feet, &c.), which it contains.

Mathematics are occupied with *quantities*, that is, with *things that can be measured*; and each branch deals with a special kind. Thus, Trigonometry primarily treats of calculations concerning lines and angles, and, in order that these may become subjects of computation, it is necessary to show how to measure them.

2. There are two methods of measuring angles :
1°. The *sexagesimal*, which is used in practical applications, such as Astronomy, Navigation, &c. 2°. The *circular* method, employed in Theoretical Trigonometry, and in the various branches of Analytical Mathematics.

DEFINITIONS.

3. The Sexagesimal Method.—In this method a right angle is divided into 90 equal parts, called *degrees*: a degree into 60 equal parts, called *minutes*: a minute into 60 equal parts, called *seconds*. These are indicated by the symbols °, ', ". Thus, $20^{\circ} 25' 30''$ denotes 20 degrees, 25 minutes, 30 seconds.

4. A third method, called the *centesimal*, was introduced in France at the time of the Revolution. In this method the right angle was divided into 100 parts, called *grades*: a grade into 100 parts, called *minutes*: a minute into 100 parts, called *seconds*. These sub-divisions are denoted by the symbols ′, ″, ‴. Thus, $32^{\prime\prime}, 14^{\prime}, 57^{\prime\prime}$. This method is now disused; but candidates at Civil Service Examinations are sometimes questioned on it.

5. Comparison of the Sexagesimal and Centesimal Measures of an Angle.

Let A denote the angle; D, G the degrees and grades respectively contained in it; then, since a right angle contains 90 degrees,

$$A : \text{a right angle} :: D : 90.$$

In like manner,

$$A : \text{a right angle} :: G : 100.$$

$$\text{Hence } D : 90 :: G : 100;$$

$$\text{therefore } D : G :: 9 : 10.$$

EXERCISES.—I.

1. Find the number of degrees in the vertical angle of an isosceles triangle, each base angle of which is double of the vertical.

Ans. 36° .

2. How many grades does a side of a regular hexagon subtend at the centre of the circumscribed circle? *Ans.* $66\frac{2}{3}^g$.

3. If m, μ denote the number of minutes in the sexagesimal and centesimal measures of an angle; prove $m : \mu :: 27 : 50$. In the same case, if s, σ denote the seconds, prove $s : \sigma :: 81 : 250$.

4. The sum of two angles is $80''$, and their difference $18'$. Find each angle.

Ans. $45', 27'$.

5. How many degrees does a side of a regular octagon subtend at any point of the circumscribed circle? *Ans.* $22\frac{1}{2}^\circ$.

6. If the number of the sides of two regular polygons be as $m : n$, and the number of grades in an angle of one equal to the number of degrees in an angle of the other; prove

$$\frac{20}{m} = \frac{18}{n} = 1.$$

7. Hence show that there are 11 pairs of regular polygons which satisfy the condition, that the number of grades in an angle of one is equal to the number of degrees in an angle of the other.

Ans. The following are the corresponding values of m and n :—5, 6; 8, 12; 10, 18; 11, 22; 12, 27; 14, 42; 15, 54; 16, 72; 17, 102; 18, 162; 19, 342.

6. The Circular Method.

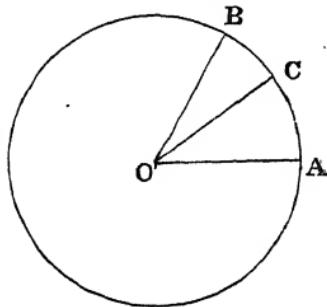
DEF. I.—*A unit circle is one whose radius is the unit of linear measure.*

DEF. II.—*The unit of circular measure is the angle at the centre of the unit circle subtended by an arc of unit length.*

7. *In a unit circle any angle at the centre is measured by the same quantity which expresses the length of the subtended arc.*

DEFINITIONS.

Dem.—Let AC be any arc, and denoting its length by a , and the corresponding central angle AOC by A ; then, if AOB be the angular unit, since angles at the centre are proportional to the arcs on which they stand (Euc. VI. xxxiii.), we have



$$AOC : AOB :: \text{arc } AC : \text{arc } AB,$$

or

$$A : 1 :: a : 1.$$

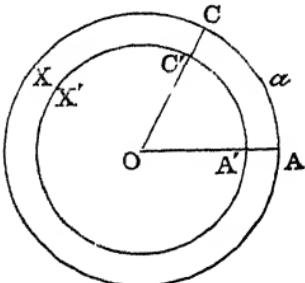
Hence

$$A = a. \quad (\text{Q. E. D.})$$

The use of the unit circle in Plane Trigonometry very much simplifies the definitions. In employing it I have followed the analogy of Spherical Trigonometry, in which the great circles are described on a unit sphere.

8. *The circular measure of the angle subtended at the centre of a circle of radius r , by an arc whose length is a , is $\frac{a}{r}$.*

Dem.—Let X, X' be two circles, whose radii are r and 1 ; $AC, A'C'$ two corresponding arcs; then, since arcs subtended by equal angles at the centres of different circles are proportional to the radii of the circles (Euc. VI. xx., Ex. 12), we have



$$OA : OA' :: AC : A'C',$$

or $r : 1 :: a : A'C'$;

therefore $A'C' = \frac{a}{r}$;

but $A'C'$ is the circular measure of the angle AOC .

Hence $\frac{a}{r}$ is its circular measure.

9. Comparison of the Sexagesimal and Circular Methods.

If π denote the ratio of the circumference of a circle to its diameter, then (Euc., note G) $\pi = 3.1415926$, and 2π will be the arithmetical measure of the circumference of the unit circle. Hence, because the circumference of a circle subtends four right angles at its centre, 2π is the circular measure of four right angles; that is, four right angles in sexagesimal measure are equal to 2π angular units.

Cor. 1.—The number of seconds in the angular unit is 206,265, nearly. For, let x denote the number of seconds in one unit of circular measure, then $2\pi x$ is the number of seconds in 2π units; that is, in four right angles. Hence

$$2\pi x = 360 \cdot 60 \cdot 60 = 1296000;$$

therefore $x = 206265$ nearly.

Cor. 2.—If a denote the length of an arc of a circle, and r the radius; then $\frac{a}{r}$ is the circular measure of the corresponding angle. Hence $(206265) \frac{a}{r}$ is the number of seconds in it.

DEFINITIONS.

EXERCISES.—II.

1. Find the circular measure of the following angles :—

1st. A right angle; 2nd, an angle of 135° ; 3rd, 45° ; 4th, 60° ; 5th, 75° ; 6th, $11^\circ 15'$; 7th, $67\frac{1}{2}^\circ$; 8th, 15° .

$$\text{Ans. } \frac{\pi}{2}; \frac{3\pi}{4}; \frac{\pi}{4}; \frac{\pi}{3}; \frac{5\pi}{12}; \frac{\pi}{16}; \frac{3\pi}{8}; \frac{\pi}{12}.$$

2. Find the circular measure—1st, of the angle of a regular pentagon; 2nd, the angle of a regular octagon; 3rd, the angle of a regular dodecagon.

$$\text{Ans. } \frac{2\pi}{5}; \frac{\pi}{4}; \frac{\pi}{6}.$$

3. Prove that the number of French minutes in the angular unit is 6366.2.

4. Prof. Airy has found that the earth's semidiameter, which is 3963 miles, subtends at the moon an angle of $57' 3'' . 16$. Find the moon's distance.

With the centre of the moon as centre, and the earth's distance as radius, describe a circle; then, if r denote the radius of this circle, we have (*Cor. 2*)

$$(206265) \frac{3963}{r} = 3423.16.$$

Hence

$$r = 238793 \text{ miles.}$$

5. The moon's diameter subtends at the earth an angle of $1868''$. Find her diameter in miles.

Let x be the moon's diameter, then we have (*Cor. 2*)

$$206265 \cdot \frac{x}{238793} = 1868.$$

Hence,

$$x = 2152 \text{ miles.}$$

6. It has been found, by the transit of Venus in 1882, that the earth's semidiameter subtends an angle of $8''.82$ at the sun. Find the sun's distance.

If x be the distance, we have (*Cor. 2*)

$$206265 \cdot \frac{3963}{x} = 8.82.$$

Hence,

$$x = 92678844 \text{ miles.}$$

7. Find the length of an arc of $10''$ on the unit circle.

$$\text{Ans. } .00004848.$$

8. The angles of a triangle are in AP , and the number of degrees in the common difference : circular measure of the greatest :: $60 : \pi$. Find the angles.

$$\text{Ans. } 30^\circ, 60^\circ, 90^\circ.$$

QUESTIONS FOR EXAMINATION.

1. What are the subject matters of Mathematics ?

Ans. Quantities, or things that can be numbered, weighed, or measured.

2. How are Mathematics divided ?

Ans. Each branch deals with a separate kind of quantity.

3. What is Trigonometry occupied with ?

Ans. Primarily with calculations relating to triangles.

4. What is meant by the numerical measure of any quantity ?

Ans. The ratio which it bears to the unit of that quantity.

5. What is meant by the unit of any quantity ?

Ans. A certain portion of it which is, by convention, adopted as the standard for measuring with.

6. What is the unit of linear measure ?

Ans. In these countries the foot ; in France the metre, &c.

7. How many methods of measuring angles are in use ?

Ans. Two : the sexagesimal, the unit of which is the degree ; and the circular method, the unit of which is called the angular unit.

8. What is meant by a unit circle ?

Ans. A circle whose radius is unity.

DEFINITIONS.

9. What is the angular unit ?

Ans. The angle subtended at the centre of the unit circle by an arc, whose length, if straightened, would be equal to the unit of linear measure.

10. What is the circular measure of any angle equal to ?

Ans. The length of the unit circle by which it is subtended.

11. If the circle be not a unit circle, what is the circular measure expressed by ?

Ans. The ratio of the length of its subtended arc to the radius of the circle.

12. What does the Greek letter π denote in Trigonometry ?

Ans. The ratio of the circumference of a circle to its diameter.

13. What is the numerical value of π ?

Ans. 3.1415926 . . .

14. What is meant by saying π is incommensurable ?

Ans. That it cannot be expressed as the ratio of any two whole numbers.

15. What special angle does $\frac{\pi}{2}$ denote ?

Ans. A right angle. The meaning is, that $\frac{\pi}{2}$ angular units is equal to a right angle.

16. In what measurement of angles is π always used ?

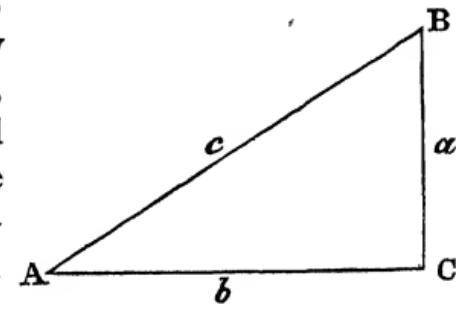
Ans. In circular measure.

CHAPTER II.

DIRECT CIRCULAR FUNCTIONS.

SECTION I.—*Circular Functions of Acute Angles.*

10. If ABC be a right-angled triangle, having the angle ACB right; then, denoting the angles by the capital letters A , B , C , respectively, and the three sides opposite these angles by the corresponding small italics, a , b , c , we have the following definitions:—



$$\text{sine } A, \quad \text{contracted into } \sin A = \frac{a}{c}. \quad (1)$$

$$\text{cosine } A, \quad , \quad \cos A = \frac{b}{c}. \quad (2)$$

$$\text{tangent } A, \quad , \quad \tan A = \frac{a}{b}. \quad (3)$$

$$\text{cotangent } A, \quad , \quad \cot A = \frac{b}{a}. \quad (4)$$

$$\text{secant } A, \quad , \quad \sec A = \frac{c}{b}. \quad (5)$$

$$\text{cosecant } A, \quad , \quad \operatorname{cosec} A = \frac{c}{a}. \quad (6)$$

The student should carefully commit to memory the foregoing equations or definitions, as upon them is founded the whole theory of Trigonometry.

EXERCISES.—III.

Calculate the circular functions, sine, cosine, &c., of the angle A in the right-angled triangles whose sides are respectively equal to— $1^{\circ} . 8, 15, 17$; $2^{\circ} . 40, 9, 41$; $3^{\circ} . 196, 315, 371$; $4^{\circ} . 480, 31, 481$; $5^{\circ} . 1700, 945, 1945$; $6^{\circ} . 440, 279, 521$; $7^{\circ} . 240, 782, 818$.

$$\text{Ans. } 1^{\circ} . \sin A = \frac{8}{17}, \tan A = \frac{8}{15}, \text{ &c.}; \quad 2^{\circ} . \sin A = \frac{40}{41}, \\ \tan A = \frac{40}{9}, \text{ &c.}; \quad 3^{\circ} . \sin A = \frac{28}{53}, \tan A = \frac{28}{45}, \text{ &c.}; \\ 4^{\circ} . \sin A = \frac{480}{481}, \tan A = \frac{31}{480}, \text{ &c.}; \quad 5^{\circ} . \sin A = \frac{340}{389}, \\ \tan A = \frac{340}{189}, \text{ &c.}; \quad 6^{\circ} . \sin A = \frac{440}{521}, \tan A = \frac{440}{279}, \text{ &c.}; \\ 7^{\circ} . \sin A = \frac{120}{409}, \tan A = \frac{120}{391}, \text{ &c.}$$

11. By adding the sum of the squares of equations (1), (2), we get

$$\sin^2 A + \cos^2 A = \frac{a^2 + b^2}{c^2} = 1. \quad (\text{EUC. I. XLVII.})$$

Hence $\sin^2 A + \cos^2 A = 1. \quad (7)$

Again, squaring equation (3), and adding 1 to both sides, we get

$$1 + \tan^2 A = 1 + \frac{a^2}{b^2} = \frac{a^2 + b^2}{b^2} = \frac{c^2}{b^2};$$

and from (5) we get $\sec^2 A = \frac{c^2}{b^2}$.

Hence $1 + \tan^2 A = \sec^2 A. \quad (8)$

In exactly the same way, from equations (4) and (6), we get

$$1 + \cot^2 A = \operatorname{cosec}^2 A. \quad (9)$$

Also, taking the products of the three pairs of equations, (1) and (6); (2) and (5); (3) and (4), we get the three equations

$$\sin A \cdot \operatorname{cosec} A = 1, \quad (10)$$

$$\cos A \cdot \sec A = 1, \quad (11)$$

$$\tan A \cdot \cot A = 1. \quad (12)$$

CIRCULAR FUNCTIONS OF ACUTE ANGLES. 11

Hence the three functions, cosec A , sec A , cot A , are the reciprocals of the three functions sin A , cos A , tan A , respectively.

Again, dividing (1) by (2), and comparing with (3), we get

$$\tan A = \frac{\sin A}{\cos A}. \quad (13)$$

$$\text{In the same manner, } \cot A = \frac{\cos A}{\sin A}. \quad (14)$$

EXERCISES.—IV.

In a right-angled triangle, given—

1. $a = \sqrt{m^2 + n^2}$, $b = \sqrt{2mn}$, calculate sin A .

$$Ans. \frac{\sqrt{m^2 + n^2}}{m + n}.$$

2. $a = 2mn$, $b = m^2 - n^2$, calculate cos A .

$$Ans. \frac{m^2 - n^2}{m^2 + n^2}.$$

3. $a = \sqrt{m^2 - 2mn}$, $b = n$, calculate sec A .

$$Ans. \frac{m - n}{n}.$$

4. $a = \sqrt{m^2 + mn}$, $c = m + n$, calculate tan A .

$$Ans. \sqrt{\frac{m^2 + mn}{mn + n^2}}.$$

5. $a = p^2 + pq$, $c = q^2 + pq$, calculate cot A .

$$Ans. \frac{\sqrt{q^2 - p^2}}{p}.$$

6. $b = lm \div n$, $c = ln \div m$, calculate cosec A .

$$Ans. \frac{n^2}{\sqrt{n^4 - m^4}}.$$

7. $\sin A = \frac{3}{5}$, $c = 200.5$, calculate a .

$$Ans. 120.3.$$

8. $\cos A = .44$, $c = 30.5$, calculate b .

$$Ans. 13.420.$$

9. $\tan A = \frac{11}{3}$, $b = \frac{27}{11}$, calculate c .

$$Ans. \frac{9\sqrt{130}}{11}.$$

DEF.—*The difference between an angle and its complement is called its complement.*

12. If in the right-angled triangle A we interchange the letters, we get the following equation :—

$$\sin B = \frac{b}{c}; \quad \text{but } \cos A = \frac{b}{c} \text{ from}$$

Hence

$$\cos A = \sin B; \quad \text{but } B \text{ is the complement of } A.$$

Therefore the cosine of an angle is equal to the sine of its complement.

Similarly, the sine of an angle is equal to the cosine of its complement.

The tangent of an angle is equal to the tangent of its complement.

The secant of an angle is equal to the secant of its complement.

The cotangent of an angle is equal to the cotangent of its complement.

The cosecant of an angle is equal to the cosecant of its complement.

EXERCISES.—V.

1. Prove $\tan A \sin A + \cos A = \sec A$.

2. " $(\sin A + \cos A) \div (\sec A + \operatorname{cosec} A)$

3. " $\cot A \cos A + \sin A = \operatorname{cosec} A$.

4. " $(\tan A - \sin A)^2 + (1 - \cos A)^2 = (1 - \sin A)^2$

5. " $(\sin A + \tan A) \div (\cot A + \operatorname{cosec} A)$

6. " $\tan A + \cot A = \sec A \operatorname{cosec} A$.

7. " $(1 - \tan A)^2 + (1 - \cot A)^2 = (\sec A - \operatorname{cosec} A)^2$

8. " $\frac{1 + \sin A}{1 + \cos A} \cdot \frac{1 + \sec A}{1 + \operatorname{cosec} A} = \tan A$.

9. " $(1 + \tan A)(1 + \cot A) = \frac{(\sin A + \cos A)^2}{\sin A \cos A}$.

10. " $(1 + \tan A)^2 + (1 + \cot A)^2 = (\sec A + \operatorname{cosec} A)^2$.

13. Graphical Method of finding all the Trigonometrical Functions when one is given.

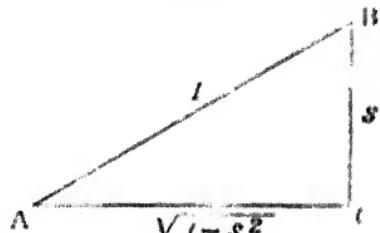
This method will be understood from the following examples:—

Ex. 1.—To express all the circular functions of an angle in terms of its sine.

Sol.—Denoting the sine by s , construct a right-angled triangle ABC , having the hypotenuse $AB = 1$, the side $BC = s$; then

$$BC \div AB = \frac{s}{1} = s;$$

but $BC \div BA = \sin A$.



Hence $\sin A = s$.

Again, $AC^2 = AB^2 - BC^2 = 1 - s^2$;

therefore $AC = \sqrt{1 - s^2}$;

but $\cos A = \frac{AC}{AB}$.

Hence $\cos A = \sqrt{1 - s^2} = \sqrt{1 - \sin^2 A}$.

Again, $\tan A = \frac{BC}{AB}$.

Hence $\tan A = \frac{s}{\sqrt{1 - s^2}} = \frac{\sin A}{\sqrt{1 - \sin^2 A}}$;

and so on for the other functions.

Ex. 2.—To express all the circular functions in terms of the tangent.

Sol.—Denoting the tangent by t , construct a right-angled triangle ABC , having the side $AC = 1$, the side $CB = t$; then

$$BC : AC = t : 1 = t;$$

but

$$BC : AC = \tan A;$$

$$\text{therefore } \tan A = t.$$

$$\text{Again, } AB^2 = AC^2 + BC^2 = 1 + t^2.$$

$$\text{Hence } AB = \sqrt{1 + t^2}.$$

$$\text{Now, } \sin A = \frac{BC}{AB};$$

$$\text{therefore } \sin A = \frac{t}{\sqrt{1 + t^2}} = \frac{\tan A}{\sqrt{1 + \tan^2 A}}.$$

Also,

$$\cos A = \frac{AC}{AB} = \frac{1}{\sqrt{1 + t^2}} = \frac{1}{\sqrt{1 + \tan^2 A}}, \text{ &c.}$$

Observation.—In these constructions it is seen that the function which is given is put as a fraction whose denominator is unity. Thus, in Ex. 1, $\sin A = \frac{s}{1}$; and in Ex. 2, $\tan A = \frac{t}{1}$.

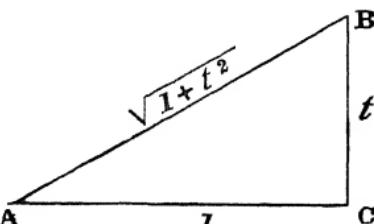
EXERCISES.—VI.

1. Given $\sin A = \frac{12}{13}$, calculate $\cos A$.

$$\text{Ans. } \cos A = \frac{5}{13}.$$

2. , , $\tan A = \frac{4}{3}$, calculate $\sin A$.

$$\text{Ans. } \sin A = \frac{4}{5}.$$



3. Given $\sin A = \frac{2mn}{m^2 + n^2}$, calculate $\tan A$.

$$\text{Ans. } \tan A = \frac{2mn}{m^2 - n^2}.$$

4. " cosec $A = 5$, calculate sec A , tan A .

$$\text{Ans. } \sec A = \frac{5}{2\sqrt{6}}; \tan A = \frac{1}{2\sqrt{6}}.$$

5. " $\sec A = \frac{41}{9}$, calculate $\sin A$, $\cot A$.

$$\text{Ans. } \sin A = \frac{40}{41}; \cot A = \frac{9}{40}.$$

6. " $\cot A = \frac{8}{15}$, calculate sec A , $\sin A$.

$$\text{Ans. } \sec A = \frac{17}{8}; \sin A = \frac{15}{17}.$$

SECTION II.—Calculation of Right-angled Triangles.

14. If the equations (1), (2) be cleared of fractions, we get $a = c \sin A$, $b = c \cos A$. (15)

Hence any side of a right-angled triangle is equal to the hypotenuse multiplied by the sine of the opposite angle, or by the cosine of the adjacent angle.

Again, from equation (3) we get

$$a = b \tan A;$$

or, by Art. 12, $a = b \cot B$. (16)

Hence, any side of a right-angled triangle is equal to the other side multiplied by the tangent of the opposite angle, or the cotangent of the adjacent angle.

15. There are four cases of right-angled triangles—

i. Given the hypotenuse and one side.

ii. " the hypotenuse and one angle.

iii. " one side and either angle.

iv. " two sides.

Case I.—Given c and a , it is required to find A .
From equation (1),

$$\sin A = \frac{a}{c}.$$

Hence A may be found from the value of $\sin A$, which is the complement of B .
 b can be calculated from the equation

$$b = \sqrt{c^2 - a^2}$$

EXERCISES.—

1. Given $c = 240$ yards, $a = 137.66$
Ans. $A =$

2. „ „ $c = 150$ yards, $a = 51.303$
Ans. $A =$

3. „ „ $c = 250$ yards, $a = 157.33$
Ans. $A =$

Case II.—Given c and A , to find a .

From Art. 14,

$$a = c \sin A, \quad b = c \cos A$$

Hence a, b are found, and the value of $\sin A$ is the complement of A .

EXERCISES.—

1. Given $c = 1760$, $A = 32^\circ$, $a =$
Ans. $a =$

2. „ „ $c = 625$, $A = 44^\circ$, $a =$
Ans. $a =$

3. „ „ $c = 300$, $A = 52^\circ$, $a =$
Ans. $a =$

CALCULATION OF RIGHT-ANGLED TRIANGLES. 17

Case III.—Given a and A , it is required to find b, c, B .

By Art. 14, $a = b \tan A$.

Hence $b = a \cot A$,

and $a = c \sin A$.

Hence $c = \frac{a}{\sin A}$,

$$B = 90^\circ - A.$$

EXERCISES.—IX.

1. Given $a = 520$, $A = 36^\circ$, find b, c, B .
Ans. $b = 715.72$; $c = 884.676$.
2. " $a = 4340$, $A = 32^\circ$, find b, c, B .
Ans. $b = 6945.45$; $c = 8189.927$.
3. " $a = 2160$, $A = 30^\circ$, find b, c, B .
Ans. $b = 3741.23$; $c = 4320$.

Case IV.—Given a and b , it is required to find c, A, B .

By Art. 14, $\tan A = \frac{a}{b}$.

Hence A can be calculated, and

$$B = 90^\circ - A,$$

and $c = \sqrt{a^2 + b^2}$.

EXERCISES.—X.

1. Given $a = 300.43$, $b = 500$, find A, B, c .
Ans. $A = 31^\circ$; $B = 59^\circ$.
2. " $a = 840.25$, $b = 1200$, find A, B, c .
Ans. $A = 35^\circ$; $B = 55^\circ$.
3. " $a = 225.1$, $b = 250$, find A, B, c .
Ans. $A = 42^\circ$; $B = 48^\circ$.

QUESTIONS FOR EXAMINATION.

1. How many parts in a triangle?

Ans. Six, namely, three sides and three angles.

2. How are they denoted by letters?

Ans. The three angles by the capital letters A , B , C , and the sides opposite them by the corresponding small letters.

3. If the triangle be right-angled, by what letters is the right angle denoted?

Ans. By the capital C , and the hypotenuse by c .

4. Name the direct circular functions of an angle.

Ans. Sine, cosine, tangent, cotangent, secant, and cosecant.

5. Name the circular functions which are reciprocals of others.

Ans. The cosecant is the reciprocal of sine, the secant of cosine, and the cotangent of tangent.

6. What is meant by the complement of an angle?

Ans. See page 12.

7. What is the relation between the functions of an angle and the functions of its complement?

Ans. The cosine of an angle is equal to the sine of its complement; the cotangent of an angle is equal to the tangent of its complement; and the cosecant equal to the secant of its complement.

8. What is the prefix 'co' in their technical terms a contraction of?

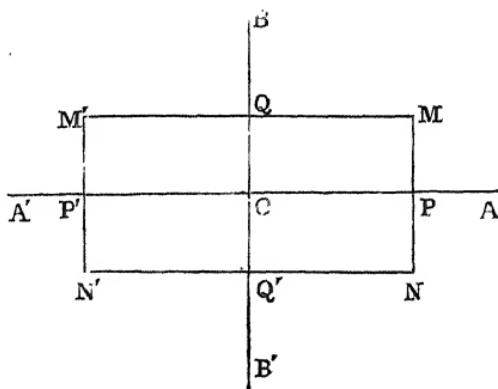
Ans. A contraction of complement. Thus, cosine means complement sine.

SECTION II.—General Definitions of Circular Functions.

16. Before commencing the general theory of circular functions, it is necessary to explain the conventional use of the signs *plus* and *minus* in Trigonometry. The application is twofold, namely, to right lines and arcs of circles.

1°. Application to right lines.

Let two perpendicular right lines AA' , BB' intersect in the point O , which is the origin from which all lines are measured; then every distance measured

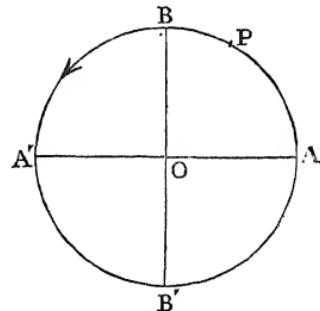


on $A'A$ to the right of O , such as OP , is positive, or $+$; those to the left, such as OP' , are negative, or $-$. Again, a line measured parallel to BB' is positive, or $+$, if it lie above $A'A$, such as PM , $P'M'$; negative, or $-$, if below, such as PN , $P'N'$.

2°. Application to arcs of circles.

Let a fixed point A on the unit circle be taken as

the origin from which all arcs are measured ; then, if we agree to consider an arc described by a variable point P , moving in the direction indicated by the arrow as positive, an arc described by a point starting from A , and moving in the opposite direction, must be regarded as negative.



17. General Expression for Arcs terminating in the same Point.

If the variable point starting from A describe the entire circumference of the unit circle n times in either direction, and afterwards describe once the arc AP , which we shall denote by θ , it is evident that the whole arc described by the point from the commencement of the motion is $2n\pi + \theta$, where n is positive or negative according to the direction of motion while the point is describing the n circumferences. Hence we have the following theorem :—
If the arc AP be denoted by θ , and n be any integer, positive or negative, $2n\pi + \theta$ is the general expression for all arcs terminating in the point P .

DEF. I.—*The four parts into which the perpendicular diameters AA' , BB' divide the circumference of a circle are called quadrants. The quadrants are named the first, second, third, fourth, respectively.*

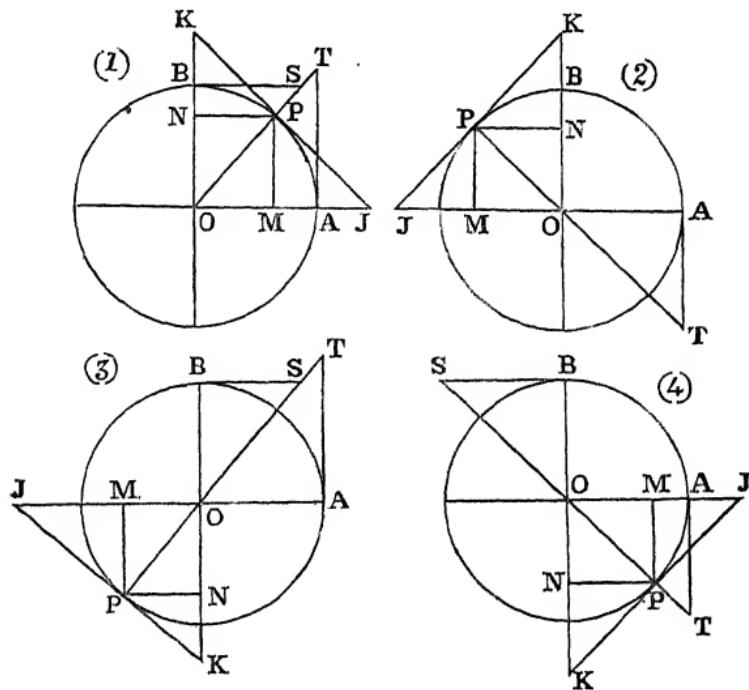
DEF. II.—*Two arcs whose sum is $\frac{\pi}{2}$ are said to be COMPLEMENTS of each other; and two arcs whose sum is π are said to be SUPPLEMENTS.*

From these definitions it follows that, if an arc be greater than a quadrant, its complement is negative;

and if greater than a semicircle, its supplement is negative.

18. Definitions of the Circular Functions.

Let OA , OB (figs. 1, 2, 3, 4) be two rectangular radii of the unit circle; P any point in the circumference; join OP , and produce it to meet the tangents to the circle at A and B in the points T and



S ; at P draw a tangent to the circle, meeting OA , OB produced in the points J , K ; lastly, from P draw PM , PN perpendicular to OA , OB . If θ denote the arc AP , then

PM is called the *sine* of the arc θ , contracted into $\sin \theta$.

OM is called the *cosine* of the arc θ , contracted into $\cos \theta$.

AT is called the *tangent* of the arc θ , contracted into $\tan \theta$.

BS is called the *cotangent* of the arc θ , contracted into $\cot \theta$.

OJ is called the *secant* of the arc θ , contracted into $\sec \theta$.

OK is called the *cosecant* of the arc θ , contracted into $\operatorname{cosec} \theta$.

MA is called the *versed sine* of the arc θ , contracted into $\operatorname{versin} \theta$.

NB is called the *coversed sine* of the arc θ , contracted into $\operatorname{coversin} \theta$.

The circular functions called versed sine and coversed sine are not much used, and the student may omit them.

19. Since (6) any arc is the measure of the corresponding central angle, the foregoing functions of the arc θ are also functions of the angle θ , and are expressed by the same notation. In the following verbal enunciations what is meant by a line is its arithmetical measure; that is, the ratio of its length to the linear unit :—

1°. *The sine of an arc is the perpendicular drawn from its extremity on the diameter which passes through the origin.*

2°. *The tangent of an arc is the line drawn touching it at the origin, and terminated by the diameter passing through its extremity.*

3°. *The secant of an arc is the portion of the diameter passing through the origin intercepted between the centre and the tangent at the extremity of the arc.*

4°. Since the lines PN or OM , BS , and OJ are respectively the sine, tangent, and secant of the arc BP , which (II. Def. II.) is the complement of AP , we see that *the cosine, the cotangent, and the cosecant of an arc are respectively the sine, the tangent, and the secant of its complement.*

It is easy to see that this method of defining the circular functions of an angle is equivalent to that of Art. 10, when the angle is acute. For if the angle POM (fig. 1) be equal to the angle BAC (fig., Art. 10), the triangles POM , BAC are equiangular. Hence $BC : AB = PM : OP$; but $PM : OP$ is the arithmetical measure of PM , since OP is the linear unit. Hence $BC : AB$ is equal to the arithmetical value of PM , and therefore both methods of defining the sine are equivalent; and the same may be shown for the other circular functions.

The method by the right-angled triangle, however, is inapplicable to any but acute angles, without an embarrassing amount of explanation.

20. Relation between the Circular Functions of an Angle.

Let θ represent the positive arc AP (figs. 1, 2, 3, 4); then we have

$$OP = 1, \sin \theta = PM, \cos \theta = OM, \sec \theta = OJ,$$

$$\operatorname{cosec} \theta = OK, \tan \theta = AT, \cot \theta = BS.$$

And since the triangle OMP is right-angled, we have

$$PM^2 + OM^2 = OP^2.$$

Hence $\sin^2 \theta + \cos^2 \theta = 1. \quad (\text{Comp. 7.}) \quad (17)$

Also, since the triangles OPJ , OPK are right-angled, and PM , PN are perpendiculars, we have (EUC. VI., VIII.)

$$ON \cdot OK = OP^2, \quad OM \cdot OJ = OP^2.$$

Therefore $\sin \theta \cdot \operatorname{cosec} \theta = 1$, (*Comp. 10.*), (18)

and $\cos \theta \cdot \sec \theta = 1$. (*Comp. 11.*). (19)

Lastly, from the pairs of similar triangles TAO , PMO ; SBO , PNO , we have

$$\frac{AT}{OA} = \frac{MP}{OP}; \quad \frac{BS}{ON} = \frac{NP}{ON};$$

therefore $\tan \theta = \frac{\sin \theta}{\cos \theta}$, (20)

and $\cot \theta = \frac{\cos \theta}{\sin \theta}$. (21)

From (20) and 21) we get

$$\tan \theta \cot \theta = 1. \quad (22)$$

From (18) we have

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta}; \quad (23)$$

$$\text{,, (19)} \quad \text{,, } \sec \theta = \frac{1}{\cos \theta}; \quad (24)$$

$$\text{,, (22)} \quad \text{,, } \cot \theta = \frac{1}{\tan \theta}. \quad (25)$$

Again, from (20), we have

$$1 + \tan^2 \theta = 1 + \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}.$$

Hence $1 + \tan^2 \theta = \sec^2 \theta$; (26)

or thus: the triangles OPJ and OAT are equal in every respect. Hence

$$OJ^2 = OT^2 = OA^2 + AT^2;$$

therefore $\sec^2 \theta = 1 + \tan^2 \theta$.

In the same manner, from the triangles OPK , OBS we have

$$OK^2 = OS^2 = OB^2 + BS^2,$$

or $\operatorname{cosec}^2 \theta = 1 + \cot^2 \theta.$ (27)

Lastly, $MA = OA - OM; NB = OB - ON,$

or $\operatorname{versin} \theta = 1 - \cos \theta;$ $\operatorname{coversin} \theta = 1 - \sin \theta.$ (28)

21. Variations of Sign and Magnitude of the Circular Functions.

1°. *Sine and Cosecant.*—From the figures 1, 2, 3, 4, of Art. 18, it is evident that the sine is positive in the first and second quadrants, and negative in the third and fourth; also that in the first quadrant it increases from 0 to 1, and in the second decreases from 1 to 0; in the third decreases from 0 to -1, and increases in the fourth from -1 to 0. Hence the greatest positive and negative values are +1 and -1 respectively. Again, since from (18)

$$\sin \theta \cdot \operatorname{cosec} \theta = 1,$$

$\operatorname{cosec} \theta$ and $\sin \theta$ will be both positive or both negative at the same time. Hence $\operatorname{cosec} \theta$ is positive in the first and second quadrants, and negative in the third and fourth.

2°. *Cosine and Secant.*—These functions are evidently positive in the first and fourth quadrants, and negative in the second and third; also the cosine varies from +1 to -1.

3°. *Tangent and Cotangent.*—Since

$$\tan \theta = \sin \theta \div \cos \theta,$$

and $\cot \theta = \cos \theta \div \sin \theta,$

it is evident that these functions will be *positive* or *negative*, according as $\sin \theta$ and $\cos \theta$ have *like* or *un-like* signs. Hence tangent and cotangent are positive both in the first and third quadrants, and negative in the second and fourth:—

Quadrant,	I.	II.	III.	IV.
Sine and Cosecant, . . .	+	+	-	-
Cosine and Secant, . . .	+	-	-	+
Tangent and Cotangent, . .	+	-	+	-

22. Periodicity of the Circular Functions.

If θ be any arc, $\pi + \theta$ is the arc which terminates at the point on the unit circle which is diametrically opposite to θ . Hence from the diagram we see that

$$\sin(\pi + \theta) = -\sin \theta; \quad (29)$$

$$\cos(\pi + \theta) = -\cos \theta; \quad (30)$$

therefore $\tan(\pi + \theta) = \tan \theta, \quad (31)$

and $\cot(\pi + \theta) = \cot \theta. \quad (32)$

The functions $\tan \theta$, $\cot \theta$, therefore, do not alter when θ is increased by π ; or in other words, *the circular functions* $\tan \theta$, $\cot \theta$ *are periodic, and the amplitude of their period is π .* Again, in like manner, we may increase the arc $\pi + \theta$ by π , without altering either the tangent or cotangent, and so on as often as we please. Hence

$$\tan \theta = \tan(n\pi + \theta), \quad (33)$$

$$\cot \theta = \cot(n\pi + \theta), \quad (34)$$

where n denotes any integer, either positive or negative.

Again, the sin, cos, sec, and cosec of an arc depend only on the positions of its origin and extremity; and it has been seen (Art. 17) that $2n\pi + \theta$ is the general expression for all arcs terminating in the same point. Hence

$$\sin \theta = \sin (2n\pi + \theta), \quad (35)$$

$$\cos \theta = \cos (2n\pi + \theta), \quad (36)$$

$$\sec \theta = \sec (2n\pi + \theta), \quad (37)$$

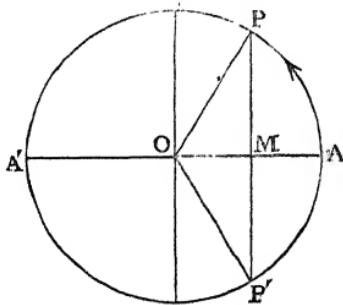
$$\operatorname{cosec} \theta = \operatorname{cosec} (2n\pi + \theta), \quad (38)$$

where n denotes any integer, positive or negative.

Hence the four functions, sin, cos, sec, cosec, are periodic, and the amplitude of each period is 2π .

23. Circular Functions of Negative Angles.

Let the arc AP on the unit circle be denoted by θ ; then, if the line PM be produced until it meet the circle again on the negative side of AA' in P' , it is evident that the angle AOP' is equal to AOP ; but, being measured in the opposite direction, it has (Art. 16, 2°) a contrary sign. Hence it must be denoted by $-\theta$.



Again, since MP , MP' are measured in opposite directions, they have contrary signs. Hence

$$MP' = -MP;$$

$$\text{but } MP = \sin \theta, \quad MP' = \sin (-\theta);$$

therefore $\sin(-\theta) = -\sin\theta.$ (39)

Hence, if two arcs or angles be equal in magnitude, but have contrary signs, their sines are equal in magnitude, and have opposite signs.

Again, the line OM is the cosine of the arc $AP,$ and also the cosine of the arc $AP'.$ Hence

$$\cos(-\theta) = \cos\theta. \quad (40)$$

Therefore, if two arcs differ only in sign, their cosines are identical.

From the equations (39), (40) we infer also that the tangent, cotangent, and cosecant change signs when the angle changes, but that the secant remains unaltered.

24. Circular Functions of Supplemental Angles.

Since $\sin(\pi + \theta) = -\sin\theta$ (equation 29),

then, changing θ into $-\theta,$ we get, from Art. 23,

$$\sin(\pi - \theta) = \sin\theta; \quad (41)$$

but $\pi - \theta$ and θ are supplements, since their sum is $\pi.$ Hence the sines, and therefore the cosecants, of two supplemental angles are equal.

Again, from equation (30), we have

$$\cos(\pi + \theta) = -\cos\theta;$$

and changing θ into $-\theta,$ we get, from Art. 23,

$$\cos(\pi - \theta) = -\cos\theta. \quad (42)$$

Hence the cosines, and therefore the secants, of two supplemental angles are equal in magnitude, but have contrary signs.

Cor. 1.—By dividing equations (41), (42) by each other, we see that the tangents, and therefore the cotangents, of two supplemental angles are equal in magnitude, but have contrary signs.

Cor. 2.—From equations (29), (41), we infer that

$$\sin(\pi \pm \theta) = \mp \sin \theta;$$

and since the sine is a periodic function with an amplitude 2π , we get

$$\sin\{(2n+1)\pi \pm \theta\} = \mp \sin \theta. \quad (43)$$

In like manner,

$$\cos\{(2n+1)\pi \pm \theta\} = -\cos \theta, \quad (44)$$

and $\tan(n\pi \pm \theta) = \pm \tan \theta. \quad (45)$

EXERCISES.—XI.

Prove the following identities :—

1. $\frac{1}{\cot \theta} \cdot \frac{1}{\sec \theta} = \sin \theta.$

2. $\cot \theta \cdot \frac{1}{\cosec \theta} = \cos \theta.$

3. $\sin^2 \theta + \cosec^2 \theta = \sin^4 \theta.$

4. $\cosec \theta + \cot \theta = \sec \theta.$

5. $\sec^4 \theta - \tan^4 \theta = \sec^2 \theta + \tan^2 \theta.$

$$6. \cot^2 A - \cot^2 B = \frac{\sin^2 B - \sin^2 A}{\sin^2 A \cdot \sin^2 B}.$$

$$7. (\cot \theta + \operatorname{cosec} \theta)^2 = \frac{1 + \cos \theta}{1 - \cos \theta}.$$

$$8. \operatorname{cosec}^4 \theta + \cot^4 \theta = 1 + 2 \cot^2 \theta \cdot \operatorname{cosec}^2 \theta.$$

$$9. \sin \theta (1 + \tan \theta) + \cos \theta (1 + \cot \theta) = \sec \theta + \operatorname{cosec} \theta.$$

$$10. \sin \theta (\cos \theta + \sin \theta \cdot \tan \theta) + \cos \theta (\sin \theta + \cos \theta \cdot \tan \theta) \\ = \tan \theta + \cot \theta.$$

$$11. 1 + 2(\sin^6 \theta + \cos^6 \theta) = 3(\sin^4 \theta + \cos^4 \theta).$$

$$12. \sin \theta = \frac{\tan \theta}{\sqrt{1 + \tan^2 \theta}} = \frac{1}{\sqrt{1 + \cot^2 \theta}} = \frac{\sqrt{\sec^2 \theta - 1}}{\sec \theta}.$$

$$13. \cos \theta = \frac{\cot \theta}{\sqrt{1 + \cot^2 \theta}} = \frac{1}{\sqrt{1 + \tan^2 \theta}} = \frac{\sqrt{\operatorname{cosec}^2 \theta - 1}}{\operatorname{cosec} \theta}.$$

$$14. \tan \theta = \frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}} = \frac{\sqrt{1 - \cos^2 \theta}}{\cos \theta} = \frac{1}{\sqrt{\operatorname{cosec}^2 \theta - 1}}.$$

15. Prove by a construction that the secant of an angle in the first quadrant is greater than the tangent, and the cosecant greater than the cotangent.

16. Construct an angle— 1° , whose cosecant is 2; 2° , whose cosine is equal to half the sine; 3° , whose sine is $\frac{3}{8}$ of the tangent.

17. Prove, by means of a construction, that $2 \sin \alpha$ is greater than $\sin 2\alpha$.

18. What must be the magnitude of α , if $\sin \alpha$ be half the side of a regular polygon of n sides inscribed in the unit circle?

19. Prove, by means of a construction—

$$1^\circ. \quad (\tan \alpha - \sin \alpha)^2 + (1 - \cos \alpha)^2 = (\sec \alpha - 1)^2;$$

$$2^\circ. \quad (\operatorname{cosec} \alpha - \sec \alpha)^2 = (1 - \tan \alpha)^2 + (\cot \alpha - 1)^2.$$

QUESTIONS FOR EXAMINATION.

1. What functions decrease in the first quadrant while the angle increases?

Ans. The cos, cosec, cot.

2. What functions decrease in the second quadrant (in absolute magnitude, without respect to sign) while the angle increases?

Ans. The sin, tan, sec.

3. Name the circular functions that are positive in the second, third, and fourth quadrants respectively.

Ans. See page 26.

4. What functions have the same values for negative angles as for the corresponding positive angles?

Ans. The cos and sec.

5. What functions have the same values for an angle and its supplement? What functions have the same values with contrary signs?

Ans. 1°. The sin and cosec; 2°. cos, sec, tan, and cot.

6. State the relations between the circular functions of two angles whose sum is four right angles.

Ans. They are equal.

7. What is meant by the statement, that circular functions are periodic?

Ans. That their values remain unaltered if the angle be increased by a constant, called the period.

8. What circular functions have a period of π ?

Ans. The tan and cot.

9. What circular functions have a period of 2π ?

Ans. The sin, cos, sec, cosec.

10. Simplify the following expressions :—

$$1^{\circ}. \quad a \cos(90 - \alpha) + b \cos(90 + \alpha).$$

$$2^{\circ}. \quad \sin(90 - \alpha) \cos(90 - \alpha).$$

$$3^{\circ}. \quad (a - b) \tan(90 - \alpha) + (a + b) \cot(90 + \alpha).$$

$$4^{\circ}. \quad \frac{\sin \alpha \cdot \tan(180 + \alpha)}{\tan \alpha \cdot \cos(90 - \alpha)}.$$

$$5^{\circ}. \quad \frac{(a^2 - b^2) \cot(\pi - \alpha)}{\cos(\pi + \alpha)} + \frac{(a^2 + b^2) \tan\left(\frac{\pi}{2} - \alpha\right)}{\cot(\pi - \alpha)}.$$

$$6^{\circ}. \quad \frac{\sin\left(\frac{\pi}{2} + \alpha\right) \cos\left(\frac{\pi}{2} - \alpha\right)}{\cos(\pi + \alpha)} + \frac{\sin(\pi - \alpha) \cdot \cos\left(\frac{\pi}{2} + \alpha\right)}{\sin(\pi + \alpha)}.$$

11. Transform the following expressions into others which will contain only $\sin \alpha$:—

$$1^{\circ}. \quad \sin \alpha \cos^2 \alpha + \frac{\tan \alpha}{\cos \alpha} - \frac{\cot \alpha}{\cos \alpha \cdot \sin \alpha}.$$

$$2^{\circ}. \quad \operatorname{cosec} \alpha \cdot \sin \alpha + \frac{\cos \alpha}{\cot \alpha} - \frac{\sin \alpha}{\cos \alpha}.$$

$$3^{\circ}. \quad \tan^2 \alpha + 1 - \operatorname{cosec}^2 \alpha + 2 \sin \alpha \cos \alpha.$$

$$4^{\circ}. \quad \sec^2 \alpha - \tan^2 \alpha + \cos^2 \alpha - \cot^2 \alpha.$$

12. Calculate the functions of the angle θ from the following equations :—

$$1^{\circ}. \quad 9 \sin^2 \theta + 27 \sin \theta = 10.$$

$$\text{Ans. } \sin \theta = \frac{1}{3}.$$

$$2^{\circ}. \quad \cos \theta = 2 - 3 \cos^2 \theta.$$

$$\text{Ans. } \cos \theta = \frac{2}{3}.$$

$$3^{\circ}. \quad 2 \tan \theta + \frac{1}{2} \cot \theta = 2 \cot \theta.$$

$$\text{Ans. } \tan \theta = \frac{\sqrt{3}}{2}.$$

$$4^{\circ}. \quad \frac{2 \tan \theta + 1}{2 \tan \theta - 1} + \frac{2 \tan \theta - 1}{2 \tan \theta + 1} = \frac{10}{3}.$$

$$\text{Ans. } \tan \theta = \pm 1.$$

CHAPTER III.

NATURE AND USE OF LOGARITHMS.

5. DEF.—*The logarithm (contracted log) of a number is the power to which 10, called the base, must be raised in order to produce the number. Thus :*

$$10^0 = 1, \quad \text{hence } \log 1 = 0.$$

$$10^1 = 10, \quad \text{,, } \log 10 = 1.$$

$$10^2 = 100, \quad \text{,, } \log 100 = 2.$$

$$10^3 = 1000, \quad \text{,, } \log 1000 = 3.$$

$$10^4 = 10000, \quad \text{,, } \log 10000 = 4.$$

$$10^5 = 100000, \quad \text{,, } \log 100000 = 5, \text{ &c.}$$

Theoretically, any number may be used as base; thus, if we take 3, since $3^4 = 81$, the log of 81 is 4, and this is written

$$\log_3 81 = 4,$$

the base being put as a suffix. There are, however, only two bases employed, namely, 10, in practical applications, and in theoretical mathematics the number 2.71828184, denoted by e , equal to the sum of the infinite series

$$1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{3}, \text{ &c.,}$$

which is called the Napierian base, after Napier, the inventor of logarithms.

26. Properties of Logarithms.

The principal object of logarithms is to facilitate calculation; their great utility in doing so depends on the four following propositions:—

1°. *The log of the product of two numbers is equal to the sum of their logarithms.*

For, if m, n be the numbers, x, y their logarithms, we have

$$10^x = m, \text{ hence } x = \log m.$$

$$10^y = n, \quad , \quad y = \log n.$$

Again, since $10^x = m$, and $10^y = n$, by multiplication we have

$$10^{x+y} = mn, \text{ or } x + y = \log mn;$$

$$\text{but } x + y = \log m + \log n;$$

$$\text{therefore } \log mn = \log m + \log n. \quad (46)$$

2°. *The log of the quotient of two numbers is equal to the difference of their logarithms.*

For, retaining the same notation, we have, by division,

$$10^{x-y} = \frac{m}{n};$$

$$\text{therefore } x - y = \log \frac{m}{n};$$

$$\text{but } x - y = \log m - \log n;$$

$$\text{therefore } \log \frac{m}{n} = \log m - \log n. \quad (47)$$

3°. *The log of the square root of a number is found by taking half its logarithm; the log of the cube root by taking one-third its logarithm, &c.*

For, as before, $10^x = m$;

therefore $10^{\frac{x}{2}} = m^{\frac{1}{2}}$,

or $\frac{x}{2} = \log m^{\frac{1}{2}}$;

but $\frac{x}{2} = \frac{1}{2} \log m$;

therefore $\log m^{\frac{1}{2}} = \frac{1}{2} \log m$. (48)

Similarly, $\log m^{\frac{1}{3}} = \frac{1}{3} \log m$, &c.

4°. *The log of the square of a number is equal to twice its logarithm; the log of the cube of a number is equal to three times its logarithm, &c.*

For, since $10^x = m$,

$$10^{2x} = m^2, \text{ or } 2x = \log m^2;$$

but $2x = 2 \log m$;

therefore $\log m^2 = 2 \log m$. (49)

Similarly, $\log m^3 = 3 \log m$, &c.

27. From Art. 25 we see that the log of 10 is 1, and the log of 100 is 2. Hence the log of any number between 10 and 100—that is, the log of any number consisting of two digits—is greater than 1 and less than 2—that is, it is 1 and a decimal. Thus, the log of 54 is 1.7323938. Similarly, the log of any number consisting of three digits is 2 and a decimal; of four digits, 3 and a decimal, and so on. When the log of a number consists of an integer

and a decimal, the *integer* is called the *characteristic* of the logarithm, and the *decimal* the *mantissa*. Hence the characteristic of the log of a number is known from the number of digits in the number. Thus, the characteristic of the log of 54312 is 4, because 54312 contains 5 digits.

28. Suppose we consider any number, such as 67836, we find from the tables

$$\log 67836 = 4.8314602.$$

Now, if we divide 67836 in succession by 10, 100, 1000, &c., we diminish (Art. 26, 2°) its log by 1, 2, 3, &c. Hence

$$\log 6783.6 = 3.8314602.$$

$$\log 678.36 = 2.8314602.$$

$$\log 67.836 = 1.8314602, \text{ &c.}$$

From this example we see that if two numbers differ only in the position of the decimal point, their logarithms have the same mantissa, differing only in the characteristic. We also see that the characteristic is determined by the number of digits to the left of the decimal point. Thus, the characteristic of the log of 67.836 is 1—that is, it is the same as the characteristic of the log of a number consisting of two digits. Again, if we divide 67.836 by 100, 1000, 10,000, &c., we must diminish the characteristic by 2, 3, 4, &c. Thus:

$$\log .67836 = 1.8314602.$$

$$\log .067836 = 2.8314602.$$

$$\log .0067836 = 3.8314602, \text{ &c.}$$

Here it is to be remarked that, when the characteristic of a log is negative, the sign minus is placed over it as above, and that the mantissa is always positive; and we also see that, if a number is altogether decimal, its log has a negative characteristic.

29. We will now exhibit the forms in which numbers and their logarithms are arranged in tables. In Callet's and Babbage's tables the logarithms of all numbers from 1 to 108000 are inserted.

I.—FORM FOR NUMBERS FROM 1 TO 1200.

No.	Logarithms.	No.	Logarithms.
2	3010300	1041	0174507
3	4771213	1042	0178677
4	6020600	1043	0182843

II.—FORM FOR NUMBERS FROM 1000 TO 108,000.

.	0	1	2	3	4	5	6	7	8	9
0	7466342	6420	6498	6575	6653	6731	6809	6887	6965	7042
1	7120	7198	7276	7354	7431	7509	7587	7665	7743	7821
2	7898	7976	8054	8132	8210	8287	8365	8443	8521	8598
3	8676	8754	8832	8910	8987	9065	9143	9221	9299	9376
{78, P.}	8	16	23	31	39	47	55	62	70	

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In Form I. all the mantissae are given at full length, and it is only necessary to supply the characteristic. Thus, the log of the number 1042 is 3.0178677, since the characteristic is 3.

In Form II., the first three figures of the mantissae being the same for all the numbers contained in it, they are inserted only in the first column, and the rest are supplied according to the digits placed in vertical and horizontal rows: thus, for 55827 we have corresponding to 2 in the *vertical*, and to 7 in the *horizontal* row, the number 8443, which with 746, which is common, gives .7468443 for the mantissa of 55827; and since it contains 5 figures to the left of the decimal point, the characteristic is 4. Hence

$$\log 55827 = 4.7468443.$$

The numbers in the last row are called proportional parts, and are constructed as follows:—If we take the difference between the logarithms of two consecutive numbers given in Form II., such as 55814 and 55815, we get .78; and if this be multiplied by .1, .2, .3, &c., we get the proportional parts .8, .16, .23, &c. The use of these is to find the logarithms of numbers of six or more places of figures. Thus, to find the logarithm of 558346.

Now, the log of 55834 is 4.7468987;

therefore $\log 558340$ is 5.7468987.

In like manner,

$\log 558350$ is 5.7469065.

Now, the difference of 558350 and 558340 is 10, and the difference of their logs is .78; also, the difference between the numbers 558340 and 558346 is 6; and we find the corresponding difference of logs by the proportion

$$10 : 6 :: .78 : 47,$$

and therefore 47 added to the log of 558340 gives .7469034 as the log of 558346. It will be seen that the table of proportional parts enables us to abridge the foregoing calculation.

EXERCISES.—XII.

1. Required the product of 1200 and 4.5.

$$\log(1200 \times 4.5) = \log 1200 + \log 4.5 = 3.0791812 + .6532125;$$

therefore $\log(1200 \times 4.5) = 3.7323937.$

Hence, finding in the tables the number corresponding to this log, the product of

$$1200 \times 4.5 = 5400.$$

2. Given $\log 2 = 3010300$, find $\log 128, 512, 3.2$.

3. " $\log 3 = .4771213$, find $\log 81, 2187, 2.43$.

4. " $\log 7 = .8450980$, find $\log 343, 2401, 16.807$.

5. Calculate the logarithms of the following numbers, making use of those given in Exercises 2, 3, 4:—

1°. 20736, 432, 98, 686, 1.728, .336.

2°. $42\frac{1}{2}, 686\frac{1}{4}, 63\frac{1}{4}, 392\frac{1}{4}, 882\frac{1}{16}, 1701\frac{1}{16}$.

3°. $\sqrt[4]{2}, (.03)^{\frac{1}{2}}, (.0021)^{\frac{1}{2}}, .098^3, .00042^2, .0336^{\frac{1}{2}}$.

QUESTIONS FOR EXAMINATION.

1. What is the logarithm of a number ?

2. How many systems of logarithms are in use ?

Ans. Two : common and Napierian logarithms.

3. What is the characteristic of a logarithm ? What is the mantissa ?

4. What numbers are they whose logarithms have a common mantissa ?

Ans. Numbers consisting of the same digits, and differing only in the position of the decimal point.

5. What numbers are they whose logarithms have a common characteristic ?

Ans. Those that have the same number of digits to the left of the decimal point.

6. How are roots extracted by logarithms ?

Ans. By division.

7. What numbers are they whose logarithms have negative characteristics ?

Ans. Those which are altogether decimal.

8. What part of the log of a number is never negative ?

Ans. The mantissa.

9. How is a fourth proportional to three numbers found by logarithms ?

Ans. By adding the logs of the second and third, and subtracting the log of the first.

10. Upon what principle does the theory of proportional parts depend ?

Ans. Upon the principle, that if the difference of two numbers be small compared to either of them, that difference will be nearly proportional to the difference of their logarithms.

11. If every log in the tables were doubled, the results would be the logs of the same numbers, but with a different base; prove this, and mention what base.
12. If, instead of being doubled, they were halved, what would be the base?
13. If a and b be any two numbers; prove

$$a^{\log b} = b^{\log a}.$$

14. How many figures in 35^{99} ?

Ans. 153.

15. If .005 be raised to the 12th power, how many cyphers will it commence with?

Ans. 27.

30. Right-angled Triangles calculated by Logarithms.

The logarithms of the circular functions, calculated to degrees and minutes for all the angles from 0° to 90° , are given in the tables, and the theory of proportional parts enables us, by means of these, to calculate them to seconds. It is easy to see that it would be sufficient to give them from 0° to 45° , since any function of an angle between 45° and 90° is equal to another function of an angle between 0° and 45° : thus

$$\sin 67^\circ = \cos 23^\circ.$$

In connexion with this remark, it is to be observed that angles greater than 45° are given at the foot of the page in the tables, and those less than 45° at the top.

2° . Since some of the circular functions are less than unity for all angles less than 90° , their loga-

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rithms have negative characteristics : thus

$$\sin 30^\circ = .5,$$

and $\log \sin 30^\circ = \log .5 = -1.6989700,$

and so on for others. In order to avoid this inconvenience, the characteristic of the logarithm of each circular function is increased by 10, and the logarithms so increased are those registered in the tables. These, called tabular logarithms, are distinguished by the word log placed before them being written with a capital letter, thus

$$\text{Log } \sin 30 = 9.6989700.$$

The following exercises will familiarize the student with the use of logarithms.

Case I.—Given a and b , it is required to find A , B , and c .

By equation (3),

$$\tan A = \frac{a}{b}.$$

Hence $\text{Log } \tan A - 10 = \log a - \log b;$

therefore $\text{Log } \tan A = 10 + \log a - \log b. \quad (50)$

Again, equation (15),

$$c \sin A = a.$$

Hence $\log c + \text{Log } \sin A - 10 = \log a;$

therefore $\log c = 10 + \log a - \text{Log } \sin A. \quad (51)$

EXAMPLE.

Given $a = 525.1$, $b = 785.3$, find A, B, c .

$$\begin{array}{rcl} 10 + \log a & = 12.7202420 \\ \log b & = 2.8950356 \end{array}$$

$$\text{therefore } \begin{array}{rcl} \text{Log tan } A & = 9.8252064 \\ \text{Log tan } 33^\circ .46 & = 9.8251660 \end{array}$$

$$\frac{404 \times 60}{2734} = 9''.$$

Tabular diff. = 2704;

$$\text{therefore } A = 33^\circ 46' 9'',$$

$$B = 56^\circ 13' 51''.$$

To find c —

$$\begin{array}{rcl} 10 + \log a & = 12.7202420 \\ \text{Log sin } 33^\circ 46' 9'' & = 9.7449563 \end{array}$$

$$\begin{array}{rcl} \log c & = 2.9752857 \\ \log 944.68 & = 2.9752847 \end{array}$$

10.

Proportional part 2.

$$\text{Hence } c = 944.682.$$

EXERCISES.—XIII.

1. Given $a = 244$, $b = 151$, calculate A, B, c .
2. , , $a = 2.0054$, $b = 151$, calculate A, B, c .
3. , , $a = 12$, $b = 16$, calculate A, B, c .
4. , , $a = 12.291$, $b = 149.50$, calculate A, B, c .
5. , , $a = 46.339$, $b = 3.8641$, calculate A, B, c .

Case II.—Given a and c , it is required to find A, B, b .

From equation (51),

$$\log \sin A = 10 + \log a - \log c. \quad (52)$$

Again (Euc. I. XLVII.),

$$b^2 = c^2 - a^2 = (c + a)(c - a).$$

$$\text{Hence } 2 \log b = \log(c + a) + \log(c - a). \quad (53)$$

EXAMPLE.

Given $a = 4754$, $c = 5850$; calculate A, B, b .

$$\begin{array}{r} 10 + \log a = 13.6770592 \\ \log c = 3.7671559 \\ \hline \end{array}$$

$$\begin{array}{r} \log \sin A = 9.9099033 \\ \log \sin 54^\circ 21' = 9.9098728 \\ \hline \end{array}$$

$$\frac{305 \times 60}{907} = 20''.$$

Tabular diff. = 907.

Hence

$$A = 54^\circ 21' 20'',$$

$$B = 35^\circ 38' 40''.$$

$$\log(c + a) = 4.0254697$$

$$\log(c - a) = 3.0398106$$

$$\begin{array}{r} 2) 7.0652803 \\ \hline \end{array}$$

$$\log b = 3.5326401, \quad b = 3409.1.$$

EXERCISES.—XIV.

1. Given $a = 1758$, $c = 2194$, calculate A, B, b .

2. ,,, $a = 338$, $c = 379$, calculate A, B, b .

3. Given $a = 4476$, $c = 8590$, calculate A , B , b .

4. "", $a = 7178$, $c = 8653$, calculate A , B , b .

5. "", $a = 792$, $c = 1734$, calculate A , B , b .

Case III.—Given a and A , it is required to find b , c , B .

From equation (50),

$$\text{Log tan } A = 10 + \log a - \log b;$$

$$\text{therefore } \log b = 10 + \log a - \text{Log tan } A. \quad (54)$$

And from (52),

$$\log c = 10 + \log a - \text{Log sin } A. \quad (55)$$

EXERCISES XV.

1. Given $a = 849$, $A = 65^\circ 14' 0''$, calculate b , c , B .

2. "", $a = 7178$, $A = 56^\circ 3' 0''$, calculate b , c , B .

3. "", $a = 174$, $A = 20^\circ 9' 17''$, calculate b , c , B .

4. "", $a = 638$, $A = 43^\circ 22' 26''$, calculate b , c , B .

5. "", $a = 0.87807$, $A = 31^\circ 0' 26''$, calculate b , c , B .

Case IV.—Given c and A , it is required to find a , b , B .

By taking the logarithms of equations (15), (16), we get

$$\log a = \log c + \text{Log sin } A - 10, \quad (56)$$

$$\log b = \log c + \text{Log cos } A - 10. \quad (57)$$

EXERCISES.—XVI.

1. Given $c = 205$, $A = 41^\circ 11' 17''$, calculate a, b, B .
2. " $c = 1734$, $A = 27^\circ 10' 35''$, calculate a, b, B .
3. " $c = 934$, $A = 32^\circ 52' 35''$, calculate a, b, B .
4. " $c = 214.259$, $A = 26^\circ 52' 14''$, calculate a, b, B .
5. " $c = 210.11$, $A = 22^\circ 36' 38''$, calculate a, b, B .

Observation.—When a sought angle differs but little from 90° , the calculation by means of its sine will not be very accurate, because the sines of angles near 90° have scarcely any differences, while the angles themselves may have a sensible difference. Similarly, if an angle be small, it is better not to calculate it by its cosine.

QUESTIONS FOR EXAMINATION.

1. Why is the characteristic in the tabular logarithms of the circular functions too great by 10?
2. When an angle is greater than 45° its Log tan is greater than 10, and its Log cot less than 10. Why?
3. Why are the characteristics of the Log-secants and Log-cosecants of all angles never less than 10?
4. When cannot an angle be found very accurately by means of its sine?
5. What angles are to be found in the tables at the foot of the page?

CHAPTER IV.

FUNCTIONS OF THE SUMS AND DIFFERENCES OF ANGLES.

1. LEMMA.—*In any triangle ABC, any side (BC) divided by the diameter of the circumcircle is equal to the sine of the opposite angle (A).*

Dem. — Draw the diameter BD . Join CD ; then, because BD is the diameter, the angle BCD is right. Hence

$$BC : BD = \sin BDC.$$

That is (EUC. III. xxii.) $= \sin BAC$. Hence, denoting the diameter BD by δ , the side BC by a , and the opposite angle by A , we have

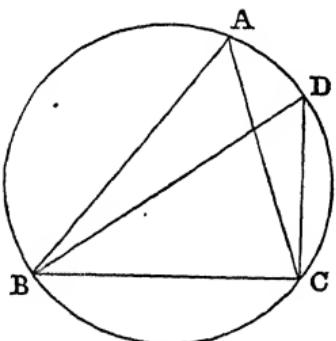
$$\frac{a}{\delta} = \sin A. \quad (58)$$

Cor. — In any triangle,

$$a : b :: \sin A : \sin B. \quad (59)$$

For $a : \delta = \sin A$, $b : \delta = \sin B$.

32. *The sine of the sum of two angles (A, B) is equal to the sine of the first multiplied by the cosine of the second, plus the cosine of the first multiplied by the sine of the second.*



Dem.—Let ON be a radius of the unit-circle; and let the angles QON , NOP be denoted by α , β , respectively; let fall the perpendiculars NP , NQ . Join PQ ; then, since the points O , P , N , Q are concyclic by Ptolemy's theorem (Euc. p. 232), we have

$$PQ \cdot ON = QN \cdot OP + OQ \cdot NP;$$

but ON is unity. Hence, by the lemma,

$$PQ = \sin QOP = \sin(\alpha + \beta).$$

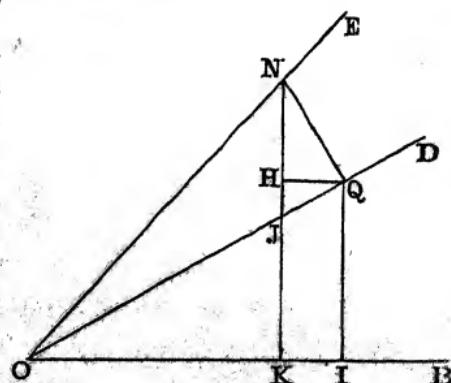
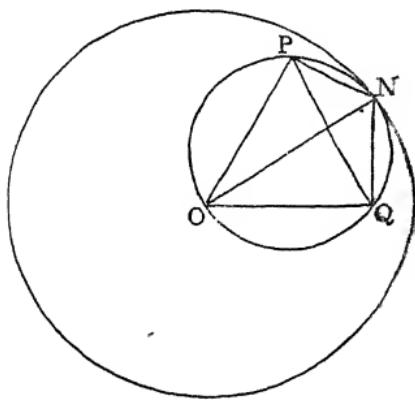
Also

$$QN = \sin \alpha, \quad NP = \sin \beta, \quad OQ = \cos \alpha, \quad OP = \cos \beta.$$

Hence

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta. \quad (60)$$

Or thus:—Let the angles BOD , DOE be denoted by α , β ; then BOE will be $\alpha + \beta$. In OE take any point N . Draw NK perpendicular to OB , and NQ to OD . Draw QH parallel to OB , and QI perpendicular to OB . Now, since the triangles OKJ , JQN , are right-angled, and have the angles OJK , NJO



[S AND DIFFERENCES OF ANGLE S.

uc. I. xv.), the angle JOK is equal to
ence the angle JNQ is equal to α .

$$\begin{aligned}\sin(\alpha + \beta) &= \frac{KN}{ON} = \frac{KH + HN}{ON} = \frac{QI}{ON} + \frac{HN}{ON} \\ &= \frac{QI \cdot OQ}{OQ \cdot ON} + \frac{HN \cdot QN}{QN \cdot ON} \\ &= \sin \alpha \cos \beta + \cos \alpha \sin \beta,\end{aligned}$$

as before.

formula (60) change β into $(-\beta)$, and we
(Art. 23)

$3) = -\sin \beta$, and $\cos(-\beta) = \cos \beta$,

$\text{in } (\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta. \quad (61)$

we let α' be the complement of α , so that

$- \alpha'$; also $\sin \alpha = \cos \alpha'$; $\cos \alpha = \sin \alpha'$;

$$- \alpha' - \beta = \sin \left\{ \frac{\pi}{2} - (\alpha' + \beta) \right\} = \cos(\alpha' + \beta);$$

$$- \alpha' + \beta = \sin \left(\frac{\pi}{2} - (\alpha' - \beta) \right) = \cos(\alpha' - \beta).$$

ubstituting $\frac{\pi}{2} - \alpha'$ for α in the formulae (61)

omitting accents, we get

$$\alpha + \beta = \cos \alpha \cos \beta - \sin \alpha \sin \beta. \quad (62)$$

$$\alpha - \beta = \cos \alpha \cos \beta + \sin \alpha \sin \beta. \quad (63)$$

$$\alpha + \beta = \frac{OK}{ON} = \frac{OI - KI}{ON} = \frac{OI}{ON} - \frac{HQ}{ON}$$

$$- \frac{OQ}{ON} - \frac{HQ}{QN} \cdot \frac{QN}{ON} = \cos \alpha \cos \beta - \sin \alpha \sin \beta;$$

giving the sign of β , we get

$$\text{as } (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta.$$

EXERCISES.—XVII.

1. Given $\sin \alpha = \frac{3}{5}$, $\cos \beta = \frac{4}{5}$; find $\sin(\alpha + \beta)$, $\cos(\alpha - \beta)$.
 2. „ $\sin \alpha = \frac{1}{2}$, $\sin \beta = \frac{1}{2}$; find $\sin(\alpha - \beta)$, $\cos(\alpha + \beta)$.
 3. Prove $\sin 2\alpha = 2 \sin \alpha \cos \alpha$. (64)

In equation (60) suppose $\beta = \alpha$, and we get

$$\sin 2\alpha = \sin \alpha \cos \alpha + \cos \alpha \sin \alpha = 2 \sin \alpha \cos \alpha.$$

4. Prove $\cos 60^\circ = \frac{1}{2}$.

Since 120° and 60° are supplements, $\sin 120^\circ = \sin 60^\circ$ (Art. 24);

but $\sin 120^\circ = 2 \sin 60^\circ \cos 60^\circ$ (equation 64);

therefore $2 \sin 60^\circ \cos 60^\circ = \sin 60^\circ$.

Hence $\cos 60^\circ = \frac{1}{2}$, and therefore $\sin 60^\circ = \frac{\sqrt{3}}{2}$. (65)

5. Prove $\sin 30^\circ = \frac{1}{2}$.

Since 60° and 30° are complements, $\sin 60^\circ = \cos 30^\circ$ (Art. 12);

but $\sin 60^\circ = 2 \sin 30^\circ \cos 30^\circ$;

therefore $2 \sin 30^\circ \cos 30^\circ = \cos 30^\circ$.

Hence $\sin 30^\circ = \frac{1}{2}$, and therefore $\cos 30^\circ = \frac{\sqrt{3}}{2}$. (66)

Or thus :—

$$\cos 60^\circ = \sin 30^\circ; \text{ but } \cos 60^\circ = \frac{1}{2}; \text{ therefore } \sin 30^\circ = \frac{1}{2}.$$

6. Prove $\sin \gamma = \sin \alpha \cos(\gamma - \alpha) + \cos \alpha \sin(\gamma - \alpha)$.

7. „ $\cos \gamma = \cos \alpha \cos(\gamma - \alpha) - \sin \alpha \sin(\gamma - \alpha)$.

8. „ $\sin \gamma = \sin(\beta + \gamma) \cos \beta - \cos(\beta + \gamma) \sin \beta$.

9. „ $\cos \gamma = \cos(\beta + \gamma) \cos \beta + \sin(\beta + \gamma) \sin \beta$.

$$10. „ \sin(\alpha + \beta) \sin(\alpha - \beta) = \sin^2 \alpha - \sin^2 \beta. \quad (67)$$

$$11. „ \cos(\alpha + \beta) \cos(\alpha - \beta) = \cos^2 \alpha - \sin^2 \beta \quad (68)$$

$$= \cos^2 \beta - \sin^2 \alpha.$$

$$12. „ \frac{\sin(\alpha - \beta)}{\sin \alpha \sin \beta} + \frac{\sin(\beta - \gamma)}{\sin \beta \sin \gamma} + \frac{\sin(\gamma - \alpha)}{\sin \gamma \sin \alpha} = 0.$$

35. From the four formulae (60)–(63), called the fundamental formulae, others can be inferred by the simple processes of addition and subtraction. Thus taking the sum and the difference of (60) and (61), and also of (62) and (63), we get

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta. \quad (69)$$

$$\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos\alpha\sin\beta. \quad (70)$$

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos\alpha\cos\beta. \quad (71)$$

$$\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2\sin\alpha\sin\beta. \quad (72)$$

EXERCISES.—XVIII.

1. Prove $\sin(30^\circ + \alpha) + \sin(30^\circ - \alpha) = \cos\alpha$.
2. " $\sin 31^\circ + \sin 29^\circ = \cos 1^\circ$.
3. " $\cos(60^\circ + \alpha) + \cos(60^\circ - \alpha) = \cos\alpha$.
4. " $\sin(60^\circ + \alpha) - \sin(60^\circ - \alpha) = \sin\alpha$.
5. " $\sin 62^\circ - \sin 58^\circ = \sin 2^\circ$.
6. " $\cos(120^\circ + \alpha) + \cos(120^\circ - \alpha) + \cos\alpha = 0$.
7. " $\cos 55^\circ + \cos 65^\circ + \cos 175^\circ = 0$.
8. " $\sin(n+1)\alpha + \sin(n-1)\alpha = 2\sin n\alpha \cos\alpha$.

36. By an easy transformation the four formulae (69)–(72) give four others for the sum and the difference of the sines and the cosines of two angles. Thus, let

$$\alpha + \beta = S, \text{ and } \alpha - \beta = D;$$

then

$$\alpha = \frac{1}{2}(S+D), \quad \beta = \frac{1}{2}(S-D).$$

Substituting these values in the above formulae (69), (70), and putting, for the sake of uniformity of no-

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tation, α , β instead of S , D , we get

$$\sin \alpha + \sin \beta = 2 \sin \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha - \beta). \quad (73)$$

$$\sin \alpha - \sin \beta = 2 \cos \frac{1}{2}(\alpha + \beta) \sin \frac{1}{2}(\alpha - \beta). \quad (74)$$

$$\cos \alpha + \cos \beta = 2 \cos \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha - \beta). \quad (75)$$

$$\cos \beta - \cos \alpha = 2 \sin \frac{1}{2}(\alpha + \beta) \sin \frac{1}{2}(\alpha - \beta). \quad (76)$$

These formulae are so important, and of such frequent recurrence, that it will be necessary for the student to commit the following enunciations to memory.

Of any two angles, the

$$\text{sum of the sines} = 2 \sin \text{half sum} \cdot \cos \frac{1}{2} \text{diff. } [(73)]$$

$$\text{diff. } , , = 2 \cos \text{half sum} \cdot \sin \frac{1}{2} \text{diff. } [(74)]$$

$$\text{sum of the cosines} = 2 \cos \text{half sum} \cdot \cos \frac{1}{2} \text{diff. } [(75)]$$

$$\text{diff. } , , = 2 \sin \text{half sum} \cdot \sin \frac{1}{2} \text{diff. } [(76)]$$

The following observations on these rules are important:—
 1°. The order of the angles in formula (76) is inverted.
 2°. In each product on the right the angles are respectively the half sum and the half diff.
 3°. In the first formula there is the product of a sine and a cosine; in the second, of a cosine and a sine; in the third, of two cosines; and in the fourth the product of two sines.

EXERCISES.—XIX.

$$1. \text{ Prove } \sin 3A + \sin A = 2 \sin 2A \cos A.$$

$$2. \text{ " } \sin 3A - \sin A = 2 \cos 2A \sin A.$$

$$3. \text{ " } \cos 2A + \cos 4A = 2 \cos 3A \cos A.$$

$$4. \text{ " } \cos 4A - \cos 6A = 2 \sin 5A \sin A.$$

5. Prove $\sin 2A + \sin 2B = 2 \sin(A + B) \cos(A - B)$.
 6. „ $\sin 40^\circ + \sin 20^\circ = \cos 10^\circ$.
 7. „ $\sin 80^\circ - \sin 40^\circ = \sin 20^\circ$.
 8. „ $\cos \alpha + \cos 3\alpha + \cos 5\alpha + \cos 7\alpha = 4 \cos \alpha \cos 2\alpha \cos 4\alpha$.
 9. „ $\sin 2\alpha + \sin 4\alpha + \sin 6\alpha = \frac{\cos \alpha - \cos 7\alpha}{2 \sin \alpha}$.
 10. „ $\cos 55^\circ \cos 65^\circ + \cos 65^\circ \cos 175^\circ + \cos 175^\circ \cos 55^\circ = -\frac{1}{4}$.

37. If we take the quotient of each pair of the formulae (73)–(76), we get the following results. The first is reduced by dividing both numerator and denominator by

$$2 \cos \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha - \beta),$$

and the reduction of the others is obvious:—

$$\frac{\sin \alpha + \sin \beta}{\sin \alpha - \sin \beta} = \frac{\tan \frac{1}{2}(\alpha + \beta)}{\tan \frac{1}{2}(\alpha - \beta)}. \quad (77)$$

$$\frac{\sin \alpha + \sin \beta}{\cos \alpha + \cos \beta} = \tan \frac{1}{2}(\alpha + \beta). \quad (78)$$

$$\frac{\sin \alpha + \sin \beta}{\cos \beta - \cos \alpha} = \cot \frac{1}{2}(\alpha - \beta). \quad (79)$$

$$\frac{\sin \alpha - \sin \beta}{\cos \alpha + \cos \beta} = \tan \frac{1}{2}(\alpha - \beta). \quad (80)$$

$$\frac{\sin \alpha - \sin \beta}{\cos \beta - \cos \alpha} = \cot \frac{1}{2}(\alpha + \beta). \quad (81)$$

$$\frac{\cos \alpha + \cos \beta}{\cos \beta - \cos \alpha} = \cot \frac{1}{2}(\alpha + \beta) \cot \frac{1}{2}(\alpha - \beta). \quad (82)$$

EXERCISES.—XX.

1. Prove $\tan(A+B) = \frac{\cos 2B - \cos 2A}{\sin 2A - \sin 2B}.$

2. , , $\cot(A+B) = \frac{\cos 2A + \cos 2B}{\sin 2A + \sin 2B}.$

3. , , $\frac{\sin(A+B) + \sin(A-B)}{\sin(A+B) - \sin(A-B)} = \frac{\tan A}{\tan B}.$

4. , , $\frac{\cos(A+B) + \cos(A-B)}{\cos(A-B) - \cos(A+B)} = \cot A \cot B.$

38. If in the formulae (73), (75), (76) we make $\beta = 0$, and remembering that

$$\cos(0) = 1, \quad \sin(0) = 0,$$

we get $\sin a = 2 \sin \frac{1}{2} a \cos \frac{1}{2} a. \quad (83)$

$$1 + \cos a = 2 \cos^2 \frac{1}{2} a;$$

or $\cos a = 2 \cos^2 \frac{1}{2} a - 1. \quad (84)$

$$1 - \cos a = 2 \sin^2 \frac{1}{2} a;$$

or $\cos a = 1 - 2 \sin^2 \frac{1}{2} a. \quad (85)$

Subtracting (85) from (84), we get

$$\cos a = \cos^2 \frac{1}{2} a - \sin^2 \frac{1}{2} a. \quad (86)$$

The following is the enunciation of these theorems:—

sine of an angle = $2 \sin \frac{1}{2} \text{angle} \cdot \cos \frac{1}{2} \text{angle}.$

$1 + \cos \text{angle} = 2 \cos^2 \frac{1}{2} \text{angle}.$

$1 - \cos \text{angle} = 2 \sin^2 \frac{1}{2} \text{angle}.$

$\cos \text{of angle} = \cos^2 \frac{1}{2} \text{angle} - \sin^2 \frac{1}{2} \text{angle}.$

Cor. 1.— $\sin 2\alpha = 2 \sin \alpha \cos \alpha.$

(Compare equation (64)).

Cor. 2.—

$$\cos 2\alpha = 2 \cos^2 \alpha - 1 = \cos^2 \alpha - \sin^2 \alpha = 1 - 2 \sin^2 \alpha. \quad (87)$$

These are obtained from (83), (84), (85), (86), by changing $\frac{1}{2}\alpha$ into α . They may also be proved by supposing $\alpha = \beta$ in the formulae (60), (62).

Cor. 3.—

$$\sin(\alpha + \beta) = 2 \sin \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha + \beta). \quad (88)$$

EXERCISES.—XXI.

1. Prove $\sin 45^\circ = \frac{1}{\sqrt{2}}$. In equation (85) put $\alpha = 90^\circ$, and

we get $2 \sin^2 45^\circ = 1.$

Hence $\sin 45^\circ = \frac{1}{\sqrt{2}}. \quad (89)$

2. Prove $\cos 45^\circ = \frac{1}{\sqrt{2}}$. (Make use of equation (84)). (90)

3. " $\frac{1 - \cos \alpha}{\sin \alpha} = \tan \frac{1}{2} \alpha. \quad (91)$
 (Make use of equations (83), (85)).

4. " $\frac{\sin \alpha}{1 + \cos \alpha} = \tan \frac{1}{2} \alpha. \quad (92)$

5. " $\frac{1 - \cos \alpha}{1 + \cos \alpha} = \tan^2 \frac{1}{2} \alpha. \quad (93)$

6. " $\operatorname{cosec} \alpha - \cot \alpha = \tan \frac{1}{2} \alpha.$

7. " $\operatorname{cosec} \alpha + \cot \alpha = \cot \frac{1}{2} \alpha.$

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8. Prove $2 \cot \alpha = \cot \frac{1}{2}\alpha - \tan \frac{1}{2}\alpha$.

9. , , $\tan \alpha + 2 \tan 2\alpha + 4 \tan 4\alpha = \cot \alpha - 8 \cot 8\alpha$.

10. , , $\frac{\sin \alpha + \sin \beta}{\sin (\alpha + \beta)} = \frac{\cos \frac{1}{2}(\alpha - \beta)}{\cos \frac{1}{2}(\alpha + \beta)}$. (94)

11. , , $\frac{\sin \alpha - \sin \beta}{\sin (\alpha + \beta)} = \frac{\sin \frac{1}{2}(\alpha - \beta)}{\sin \frac{1}{2}(\alpha + \beta)}$. (95)

39. If we divide formula (60) by (62), and reduce the right-hand side by dividing both numerator and denominator by $\cos \alpha \cos \beta$, we get

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}. \quad (96)$$

In like manner, from (61) and (63) we get

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}. \quad (97)$$

In (96) put $\alpha = \beta$, and we get

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}. \quad (98)$$

EXERCISES.—XXII.

1. Prove $\tan 45^\circ = 1$. (99)

$$\text{, , } \tan 45^\circ = \sin 45^\circ \div \cos 45^\circ = \frac{1}{\sqrt{2}} \div \frac{1}{\sqrt{2}}$$

2. If $\tan \alpha = \frac{1}{2}$, $\tan \beta = \frac{1}{3}$; prove $\alpha + \beta = 45^\circ$.

3. If $\tan \alpha = \frac{1}{3}$, $\tan \beta = \frac{1}{4}$; prove $\alpha + \beta = 45^\circ$.

4. Prove $\tan \alpha + \tan \beta = \frac{\sin(\alpha + \beta)}{\cos \alpha \cos \beta}$.

5. , , $\tan \alpha - \tan \beta = \frac{\sin(\alpha - \beta)}{\cos \alpha \cos \beta}$.

6. Prove $\tan(\alpha + 45) = \frac{1 + \tan \alpha}{1 - \tan \alpha}.$

7. " $\tan(\alpha - 45) = \frac{\tan \alpha - 1}{\tan \alpha + 1}.$

8. " $\frac{\tan \alpha + \tan \beta}{\tan \alpha - \tan \beta} = \frac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)}.$

9. " $\tan^2 \alpha - \tan^2 \beta = \frac{\sin^2 \alpha - \sin^2 \beta}{\cos^2 \alpha \cdot \cos^2 \beta}.$

10. " $\frac{\cot \alpha - \tan \alpha}{\cot \alpha + \tan \alpha} = \cos 2\alpha.$

40. Formulae for the Sum of three or more Angles.

Let α, β, γ be three angles ; then

$$\sin(\alpha + \beta + \gamma) = \sin(\alpha + \beta) \cos \gamma + \cos(\alpha + \beta) \sin \gamma;$$

and substituting, from (60) and (62), their values for

$$\sin(\alpha + \beta), \quad \cos(\alpha + \beta),$$

we get

$$\begin{aligned} \sin(\alpha + \beta + \gamma) &= \sin \alpha \cos \beta \cos \gamma + \sin \beta \cos \gamma \cos \alpha \\ &\quad + \sin \gamma \cos \alpha \cos \beta - \sin \alpha \sin \beta \sin \gamma. \end{aligned} \quad (100)$$

In like manner,

$$\begin{aligned} \cos(\alpha + \beta + \gamma) &= \cos \alpha \cos \beta \cos \gamma - \cos \alpha \sin \beta \sin \gamma \\ &\quad - \cos \beta \sin \gamma \sin \alpha - \cos \gamma \sin \alpha \sin \beta. \end{aligned} \quad (101)$$

In these formulae, if we make β, γ each equal to α , we get

$$\sin 3\alpha = 3 \sin \alpha \cos^2 \alpha - \sin^3 \alpha,$$

$$\cos 3\alpha = \cos^3 \alpha - 3 \sin^2 \alpha \cos \alpha.$$

and replacing in the former $\cos^2 \alpha$ by $1 - \sin^2 \alpha$ in the latter, $\sin^2 \alpha$ by $1 - \cos^2 \alpha$, we get

$$\sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha. \quad (1)$$

$$\cos 3\alpha = 4 \cos^3 \alpha - 3 \cos \alpha. \quad (1)$$

41. If we divide (100) by (101), and reducing the numerator and the denominator on the right-hand side by $\cos \alpha \cos \beta \cos \gamma$, we get

$$\tan(\alpha + \beta + \gamma) = \frac{\tan \alpha + \tan \beta + \tan \gamma - \tan \alpha \tan \beta \tan \gamma}{1 - \tan \alpha \tan \beta - \tan \beta \tan \gamma - \tan \gamma \tan \alpha} \quad (1)$$

In this result, if we make β, γ each equal to get

$$\tan 3\alpha = \frac{3 \tan \alpha - \tan^3 \alpha}{1 - 3 \tan^2 \alpha}. \quad (1)$$

Cor. 1.—If $\alpha + \beta + \gamma = m\pi$, where m is any integer, then $\tan \alpha + \tan \beta + \tan \gamma = \tan \alpha \tan \beta \tan \gamma$. (1)

Cor. 2.—If $\alpha + \beta + \gamma = \frac{\pi}{2}$, $\tan \beta \tan \gamma + \tan \gamma \tan \alpha + \tan \alpha \tan \beta = 1$. (1)

EXERCISES.—XXIII.

1. Find $\sin 18^\circ$

Putting $\alpha = 18^\circ$, we have, since

$$\sin 36^\circ = \cos 54^\circ$$

$$\sin 2\alpha = \cos 3\alpha,$$

$$\text{or } 2 \sin \alpha \cos \alpha = 4 \cos^3 \alpha - 3 \cos \alpha;$$

$$\text{therefore } 2 \sin \alpha = 4 \cos^2 \alpha - 3 = 4(1 - \sin^2 \alpha) - 3;$$

$$\text{therefore } 4 \sin^2 \alpha + 2 \sin \alpha = 1.$$

$$\text{Hence } \sin \alpha = \frac{\sqrt{5} - 1}{4}, \text{ or } \sin 18^\circ = \frac{\sqrt{5} - 1}{4}. \quad (1)$$

Or thus :— $2 \sin 36^\circ \cos 36^\circ = \sin 72^\circ,$

and $2 \sin 72^\circ \cos 72^\circ = \sin 144^\circ$ or $\sin 36^\circ.$

Hence $2 \cos 36^\circ \cos 72^\circ = \frac{1}{2};$

* but $2 \cos 36^\circ \cos 72^\circ = \cos 36^\circ + \cos 108^\circ = \cos 36^\circ - \cos 72^\circ;$

therefore $\cos 36^\circ - \cos 72^\circ = \frac{1}{2}.$

Hence, denoting $\cos 36^\circ$ and $\cos 72^\circ$ by x and y , we have

$$2xy = \frac{1}{2} \text{ and } x - y = \frac{1}{2};$$

$$\text{therefore } x = \frac{\sqrt{5} + 1}{4} \text{ and } y = \frac{\sqrt{5} - 1}{4};$$

therefore

$$\cos 36^\circ = \frac{\sqrt{5} + 1}{4}, \text{ and } \cos 72^\circ \text{ or } \sin 18^\circ = \frac{\sqrt{5} - 1}{4}. \quad (109)$$

MISCELLANEOUS EXERCISES.—XXIV.

1. Prove $\sin 2\alpha = \frac{2 \tan \alpha}{1 + \tan^2 \alpha}.$

2. , , $\cos 2\alpha = \frac{1 - \tan^2 \alpha}{1 + \tan^2 \alpha}.$

3. , $\frac{\cos \alpha - \sin \alpha}{\cos \alpha + \sin \alpha} = \sec 2\alpha - \tan 2\alpha.$

4. Prove—

$$\begin{aligned} \sin \alpha + \cos \beta &= 2 \cos \left\{ \frac{\pi}{4} - \frac{1}{2}(\alpha - \beta) \right\} \cos \left\{ \frac{\pi}{4} - \frac{1}{2}(\alpha + \beta) \right\} \\ &= 2 \sin \left\{ \frac{\pi}{4} + \frac{1}{2}(\alpha - \beta) \right\} \sin \left\{ \frac{\pi}{4} + \frac{1}{2}(\alpha + \beta) \right\}. \end{aligned}$$

5. Prove—

$$\begin{aligned} \sin \alpha - \cos \beta &= 2 \sin \left\{ \frac{\pi}{4} - \frac{1}{2}(\alpha - \beta) \right\} \sin \left\{ \frac{1}{2}(\alpha + \beta) - \frac{\pi}{4} \right\} \\ &= 2 \cos \left\{ \frac{\pi}{4} + \frac{1}{2}(\alpha - \beta) \right\} \cos \left\{ \frac{\pi}{4} + \frac{1}{2}(\alpha + \beta) \right\}. \end{aligned}$$

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6. Prove $\sec \alpha \pm \tan \alpha = \tan(45 \pm \frac{1}{2}\alpha)$.

7. " $\frac{\cos \alpha + \cos \beta}{\sin(\alpha + \beta)} = \frac{\cos \frac{1}{2}(\alpha - \beta)}{\sin \frac{1}{2}(\alpha + \beta)}$.

8. " $\frac{\cos \beta - \cos \alpha}{\sin(\alpha + \beta)} = \frac{\sin \frac{1}{2}(\alpha - \beta)}{\cos \frac{1}{2}(\alpha + \beta)}$.

9. If $\cos \theta = \frac{\cos \alpha + \cos \beta}{1 + \cos \alpha \cos \beta}$, prove $\tan^2 \frac{1}{2}\theta = \tan^2 \frac{1}{2}\alpha \tan^2 \frac{1}{2}\beta$.

10. If $\cos \theta = \frac{\cos \alpha - \cos \beta}{1 - \cos \alpha \cos \beta}$, prove $\tan^2 \frac{1}{2}\theta = \tan^2 \frac{1}{2}\alpha \cot^2 \frac{1}{2}\beta$.

11. Prove $\cos^2 \alpha = 2 \cos \left(\frac{\pi}{4} - \alpha \right) \cos \left(\frac{\pi}{4} + \alpha \right)$.

12. " $\frac{\sin \alpha + \sin 4\alpha + \sin 7\alpha}{\cos \alpha + \cos 4\alpha + \cos 7\alpha} = \tan 4\alpha$.

13. " $\frac{1 + \sin \alpha}{1 - \sin \alpha} = \tan^2 \left(\frac{\pi}{4} + \frac{\alpha}{2} \right)$.

14. " $\tan(\alpha + 60^\circ) \tan(\alpha - 60^\circ) = \frac{1 + 2 \cos 2\alpha}{1 - 2 \cos 2\alpha}$.

15. " $\cos^8 \alpha - \sin^8 \alpha = \cos 2\alpha(1 - \frac{1}{2} \sin^2 2\alpha)$.

16. " $\sin^6 \alpha + \cos^6 \alpha = (1 - \frac{3}{8} \sin^2 2\alpha)$.

17. " $\sin m\alpha = \sin \alpha \cos(m-1)\alpha + \cos \alpha \sin(m-1)\alpha$.

18. " $\cos m\alpha = \cos \alpha \cos(m-1)\alpha - \sin \alpha \sin(m-1)\alpha$.

19. " $\sin 4\alpha = 4 \sin \alpha \cos \alpha - 8 \sin^3 \alpha \cos \alpha$.

20. " $\sin 5\alpha = 5 \sin \alpha - 20 \sin^3 \alpha + 16 \sin^5 \alpha$.

21. " $\cos 4\alpha = 1 - 8 \cos^2 \alpha + 8 \cos^4 \alpha$.

22. " $\cos 5\alpha = 5 \cos \alpha - 20 \cos^3 \alpha + 16 \cos^5 \alpha$.

23. " $2 \cos^2 \alpha \cos^2 \beta - 2 \sin^2 \alpha \sin^2 \beta = \cos 2\alpha + \cos 2\beta$.

24. " $4 \sin 3\alpha \cos^3 \alpha + 4 \cos 3\alpha \sin^3 \alpha = 3 \sin 4\alpha$.

25. " $\cot \alpha - \tan \alpha = 2 \cot 2\alpha$.

26. " $\cos \alpha + \cos \beta + \cos \gamma + \cos(\alpha + \beta + \gamma)$
 $= 4 \cos \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\beta + \gamma) \cos \frac{1}{2}(\gamma + \alpha)$.

27. Prove $\cos(n+1)\cos(n-1)\alpha + \sin^2\alpha = \cos^2 n\alpha$.
 28. If A, B, C, D be four points ranged in order on a circle,
 d if $\cot AB + \cot AD = 2 \cot AC$; prove

$$\sin AB : \sin BC :: \sin AD : \sin DC.$$

29. If $\tan^2\alpha = 1 + 2\tan^2\beta$, prove $\cos^2\beta = 1 + \cos 2\alpha$.

30. Prove $\sin 6\alpha = \cos\alpha(6\sin\alpha - 32\sin^3\alpha + 32\sin^5\alpha)$.

31. . . , $\cos 6\alpha = -(1 - 18\cos^2\alpha + 48\cos^4\alpha - 32\cos^6\alpha)$.

32. " $\frac{\sin 3\alpha + \cos 3\alpha}{\sin 3\alpha - \cos 3\alpha} = \tan\left(\alpha - \frac{\pi}{4}\right) \left(\frac{1 + 2\sin^2 2\alpha}{1 - 2\sin 2\alpha} \right)$.

33. Find the circular functions of 15° .

$$\text{In } 15^\circ = \sin(45 - 30) = \sin 45 \cos 30 - \cos 45 \sin 30 = \frac{\sqrt{3} - 1}{2\sqrt{2}}$$

Similarly, $\cos 15^\circ = \frac{\sqrt{3} + 1}{2\sqrt{2}}$.

Hence

$$\tan 15^\circ = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = 2 - \sqrt{3}, \text{ and } \cot 15^\circ = 2 + \sqrt{3}.$$

34. Find the circular functions of 75° .

Since 75° is the complement of 15° , we have

$$\sin 75 = \cos 15 = \frac{\sqrt{3} + 1}{2\sqrt{2}}, \tan 75^\circ = \cot 15 = 2 + \sqrt{3}, \text{ &c.}$$

35. Having given

$$\sin \theta + \sin \phi = a, \quad \cos \theta + \cos \phi = b;$$

prove $\tan \frac{\theta}{2} + \tan \frac{\phi}{2} = \frac{4a}{a^2 + b^2 + 2b};$

" $\tan \theta + \tan \phi = \frac{8ab}{(a^2 + b^2)^2 - 4a^2};$

" $\sin(\theta + \phi) = \frac{2ab}{a^2 + b^2}.$

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36. If $\cos \theta = \frac{\cos u - e}{1 - e \cos u}$, prove $\tan \frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{u}{2}$.

37. If $\alpha = \frac{2\pi}{15}$, prove $\cos \alpha + \cos 2\alpha + \cos 4\alpha + \cos 8\alpha = \frac{1}{2}$.

38. If

$$(1 + \cos A)(1 + \cos B)(1 + \cos C) = (1 - \cos A)(1 - \cos B)(1 - \cos C),$$

prove each = $\sin A \sin B \sin C$.

39. Prove $\sin \alpha = \frac{1}{\cot \frac{1}{2}\alpha - \cot \alpha}$.

40. " $\tan \alpha = \frac{2}{\cot \frac{1}{2}\alpha - \tan \frac{1}{2}\alpha}$.

41. " $1 + \cos \alpha + \cos 2\alpha + \cos 3\alpha = 2 \cos \alpha (2 \cos^2 \alpha + \cos \alpha - 1)$.

42. " $\tan(30^\circ + \frac{1}{2}\alpha) \tan(30^\circ - \frac{1}{2}\alpha) = \frac{2 \cos \alpha - 1}{2 \cos \alpha + 1}$.

43. " $\sin 3\alpha = 4 \sin \alpha \cdot \sin(60^\circ - \alpha) \sin(60^\circ + \alpha)$.

44. " $\cos 3\alpha = 4 \cos \alpha \cdot \cos(60^\circ - \alpha) \cos(60^\circ + \alpha)$.

45. " $\sin(\alpha + \beta) \sin 3(\alpha - \beta) = \sin^2(2\alpha - \beta) - \sin^2(2\beta - \alpha)$.

46. If $\alpha + \beta + \gamma = 2\sigma$, prove $\cos 2\sigma + \cos 2(\sigma - \alpha) + \cos 2(\sigma - \beta) + \cos 2(\sigma - \gamma) = 4 \cos \alpha \cos \beta \cos \gamma$.

47. Prove $\sin \alpha + \sin 3\alpha + \sin 5\alpha \dots + \sin(2n-1)\alpha = \frac{\sin^2 n\alpha}{\sin \alpha}$.

48. " $\cos \alpha + \cos 3\alpha + \cos 5\alpha \dots + \cos(2n-1)\alpha = \frac{\sin 2n\alpha}{2 \sin \alpha}$.

CHAPTER V.

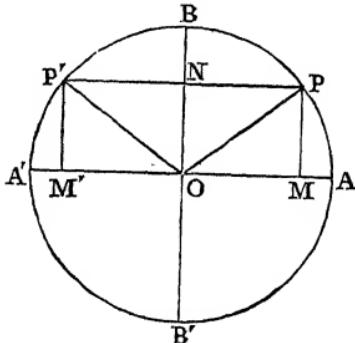
INVERSE CIRCULAR FUNCTIONS.

SECTION I.

42. When an angle is given, to each value of it there is but one value of each of the circular functions; but if the circular function be given, and it be required to find the angle, it will be seen, that to each value of the function there is an infinite number of corresponding angles. It will be sufficient to establish rules for the *sine*, *cosine*, and *tangent*; since the rules for these functions hold for their reciprocals—the *cosecant*, *secant*, and *cotangent*.

43. SINES.—*Let a be the sine of an arc; it is required to construct the arc.*

Sol.—Let O be the centre of the unit-circle, A the origin of the arcs. Draw the diameters AA' , BB' at right angles; and take on OB the line ON , whose numerical measure is equal to a . Through N draw PP' parallel to AA' ; then, since PM , $P'M'$ are each equal to ON , the sines of the arcs AP , AP' are each equal to a ; but the arcs AP , AP' are supplements; and if one of them be denoted by α , the other will



be $\pi - a$. Again, since the sine of an angle is not altered when the angle is increased by $2n\pi$ (Art. 22), it follows, if θ be the general value of the angle whose sine is a , that

$$\theta = 2n\pi + a, \quad \text{or} \quad \theta = (2n + 1)\pi - a, \quad (110)$$

where n denotes any positive or negative integer.

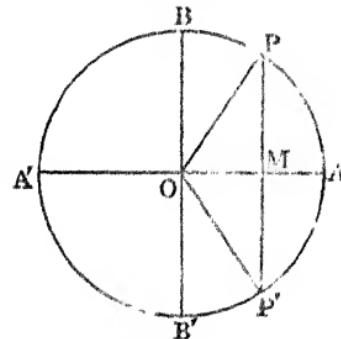
Cor.—If two angles have the same sine, either their difference is an even multiple or their sum is an odd multiple of π .

EXERCISES—XXV.

1. What is the general value of the angle whose sine is $\frac{1}{2}$?
2. What is the general value of θ if $\sin^2 \theta = \frac{1}{2}$?
3. Write down all the values of θ which satisfy the equation $\sin^2 \theta = \sin^2 a$.
4. Given $\operatorname{cosec}^2 \theta = \frac{4}{3}$, what is the general value of θ ?

44. COSINE.—*If a be the cosine of an arc, it is required to find its general value.*

Sol.—Let A be the origin of arcs; AA' , BB' two diameters at right angles to each other. Take on OA the line OM , whose numerical measure is equal to a . Through M draw PP' parallel to BB' ; then it is evident that the points P, P' will be the extremities of all arcs whose cosine is equal to a . Now, if AP be denoted by a , AP' will be (Art. 23) $-a$.



ence, if θ be the general angle whose cosine is a , we have

$$\theta = 2n\pi \pm \alpha, \quad (\text{iii})$$

where n denotes any positive or negative integer.

Cor.—If two angles have the same cosine, either their sum or their difference must be an even multiple of π .

EXERCISES.—XXVI.

1. What are the general values of θ , which satisfy

$$\sin 3\theta = \sin \theta \cos 2\theta ?$$

From the given equation we get

$$\sin(\theta + 2\theta) - \sin \theta \cos 2\theta = 0.$$

ence $\cos \theta \sin 2\theta = 0;$

therefore either $\cos \theta = 0$ or $\sin 2\theta = 0$.

From the first of these equations we get $\theta = \text{some odd multiple of } \frac{\pi}{2}$, and from the second $2\theta = \text{any multiple of } \pi$. Hence they are included in the equation $\theta = \frac{n\pi}{2}$, where n is any integer.

- 2-12. Find the general value of θ , satisfying the following equations :—

2°. $\cos \theta = \cos 2\theta.$

3°. $\cos \theta + \cos 3\theta + \cos 5\theta = 0.$

4°. $\sin \theta + \sin 2\theta + \sin 3\theta = 0.$

5°. $\sin 5\theta = 16 \sin^5 \theta.$

6°. $\sin 9\theta - \sin \theta = \sin 4\theta.$

7°. $\sin \theta + \cos \theta = \frac{1}{\sqrt{2}}.$

$$8^\circ. \quad \sin 4\theta + \sin 6\theta = 0.$$

$$9^\circ. \quad \cos \theta \cos 5\theta + \cos 5\theta \cos 7\theta = 0.$$

$$10^\circ. \quad \sec \theta = 2.$$

$$11^\circ. \quad \sin \theta - \cos \theta = 4 \sin \theta \cos^2 \theta.$$

$$12^\circ. \quad \sin 11\theta + \sin 7\theta + 2 \cos^2 \theta = 1.$$

45. TANGENT.—*If a be the tangent of an arc, it is required to find its general value.*

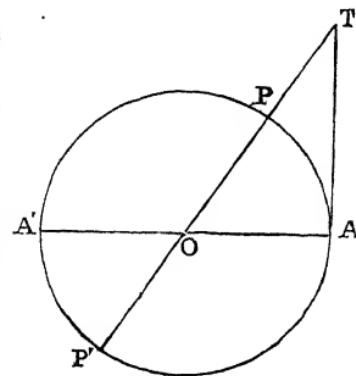
Sol.—Let APA' be the unit circle; A the origin of the arcs; O the centre.

Draw AT' touching the circle, and take AT , such that its numerical measure shall be equal to a ; join OT , cutting the circle in the points P, P' ; then any arc terminating in either of these points will have its tangent equal to a . Hence, if $AP = a$, and θ be the required arc, the general value of θ is given by the equation

$$\theta = n\pi + a, \quad (112)$$

where n denotes any positive or negative integer.

Cor.—If two arcs have the same tangent, their difference must be some multiple of π .



EXERCISES.—XXVII.

1–4. Find the general value of θ in the equations—

- 1°. $\tan \theta = 1.$
- 2°. $\tan^2 \theta = 3.$
- 3°. $\tan^2 \theta = \frac{1}{3}.$
- 4°. $\tan^2 \theta + \cot^2 \theta = 2.$

5–16. Solve the following equations :—

- 5°. $2 \sin \theta = \tan \theta.$
- 6°. $6 \cot^2 \theta - 4 \cos^2 \theta = 1.$
- 7°. $\tan \theta + \tan \left(\theta - \frac{\pi}{4} \right) = 2.$
- 8°. $\tan 2\theta + \cot \theta - 8 \cos^2 \theta = 0.$
- 9°. $\tan \theta + \cot \theta = \sqrt{2}.$
- 10°. $\cos \theta + \sqrt{3} \sin \theta = \sqrt{2}.$
- 11°. $\cot \left(\frac{\pi}{4} - \theta \right) = 3 \cot \left(\frac{\pi}{4} + \theta \right).$
- 12°. $\tan \left(\frac{\pi}{4} + \theta \right) = 1 + \sin 2\theta.$
- 13°. $\sec \theta = 2 \tan \theta.$
- 14°. $(\cot \theta - \tan \theta)^2 (2 + \sqrt{3}) = 4 (2 - \sqrt{3}).$
- 15°. $\operatorname{cosec} \theta \cot \theta = 2\sqrt{3}.$
- 16°. $\frac{\cos \theta}{1 + \sin \theta} + \tan \theta = 2.$

17. Given the difference of the lengths of the shadow of a tower, when the sun's altitudes are α° , β° , respectively = h feet; find the height of the tower.

SECTION II.

46. DEF.—The following is the notation used for inverse functions :

$$\sin^{-1}x, \cos^{-1}y, \tan^{-1}z, \&c.$$

They are read thus: The arc whose sine is x; the arc whose cosine is y; the arc whose tangent is z, &c. Thus, when we put $x = \sin \theta$, we can say conversely $\theta = \sin^{-1}x$, &c.

From the foregoing definition we see that every relation between the direct circular functions has a corresponding relation for the inverse. Thus, from the equation

$$\tan(\alpha_1 + \alpha_2) = \frac{\tan \alpha_1 + \tan \alpha_2}{1 - \tan \alpha_1 \tan \alpha_2} \quad (\text{equation (96)}),$$

we get $\alpha_1 + \alpha_2 = \tan^{-1} \left\{ \frac{\tan \alpha_1 + \tan \alpha_2}{1 - \tan \alpha_1 \tan \alpha_2} \right\}.$

Now put

$$\tan \alpha_1 = a_1, \tan \alpha_2 = a_2, \text{ then } \alpha_1 = \tan^{-1} a_1, \alpha_2 = \tan^{-1} a_2.$$

$$\text{Hence } \tan^{-1} a_1 + \tan^{-1} a_2 = \tan^{-1} \left(\frac{a_1 + a_2}{1 - a_1 a_2} \right), \quad (113)$$

which is the inverse formula corresponding to (96) for the tangent of the sum of two angles.

EXERCISES.—XXVIII.

1. Prove $2 \tan^{-1} \alpha = \tan^{-1} \frac{2\alpha}{1 - \alpha^2}.$

2. " $3 \tan^{-1} \alpha = \tan^{-1} \frac{3\alpha - \alpha^3}{1 - 3\alpha^2}.$

3. " $4 \tan^{-1} \frac{1}{5} = \tan^{-1} \frac{120}{119}.$

4. Prove $4 \tan^{-1} \frac{1}{6} - \tan^{-1} \frac{1}{7} - \tan^{-1} \frac{1}{9} = \frac{\pi}{4}$.

5. , , $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$.

6. , , $\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{9} = \frac{\pi}{4}$.

7. , , $\sin^{-1} \frac{4}{5} + \sin^{-1} \frac{5}{13} = \sin^{-1} \frac{63}{65}$.

8. , , $\sin^{-1} \frac{1}{\sqrt{82}} + \sin^{-1} \frac{4}{\sqrt{41}} = \frac{\pi}{4}$.

9. , , $\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{6} + \tan^{-1} \frac{1}{12}$, &c., to inf. = $\frac{\pi}{4}$.

10. , , if $\tan^2 \theta = \tan(\theta - \alpha) \tan(\theta - \beta)$,

$$2\theta = \tan^{-1} \frac{2 \sin \alpha \sin \beta}{\sin(\alpha + \beta)}.$$

Solve the following equations :—

11. $\cot^{-1} x + \cot^{-1}(n^2 - x + 1) = \cot^{-1}(n - 1)$.

12. $\sin^{-1} x + \sin^{-1} \frac{x}{2} = \frac{\pi}{4}$.

13. $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$.

14. $2 \tan^{-1} x = \sin^{-1} \left(\frac{2x}{1+x^2} \right) + \sin^{-1} \left(\frac{2b}{1+b^2} \right)$.

15. $\cot^{-1} \frac{I}{x+I} + \cot^{-1} \frac{I}{x-I} = \tan^{-1} 3x - \tan^{-1} x$.

SECTION III.

47. The inverse circular functions are not the only ones that have multiple values; for, as we shall see in the following propositions, the direct functions in several cases have more than one value (though not an indefinite number) when expressed in terms of each other.

48. If the cosine of an angle be given, the sine and the cosine of its half are each two-valued.

$$\text{Dem.— } 2 \sin^2 \frac{1}{2}\theta = 1 - \cos \theta \quad (85);$$

$$\text{therefore } \sin \frac{1}{2}\theta = \pm \sqrt{\frac{1 - \cos \theta}{2}}. \quad (114)$$

In like manner,

$$\cos \frac{1}{2}\theta = \pm \sqrt{\frac{1 + \cos \theta}{2}}. \quad (115)$$

49. LEMMA I.—If A lie between 45° and 225° , $\sin A - \cos A$ is positive; and if between 225° and 405° , $\sin A - \cos A$ is negative.

For,

$$\begin{aligned} \sin(A - 45^\circ) &= \sin A \cos 45^\circ - \cos A \sin 45^\circ \\ &= (\sin A - \cos A) \frac{1}{\sqrt{2}}. \end{aligned}$$

Now, if A lie between 45° and 225° , $(A - 45)$ lies between 0 and 180° , and $\sin(A - 45^\circ)$ is positive; therefore $\sin A - \cos A$ is positive. In the same manner, if A lie between 225° and 405° , $\sin A - \cos A$ is negative.

50. LEMMA II.—If A lie between -45° and 135° , $\sin A + \cos A$ is positive; and if between 135° and 315° , $\sin A + \cos A$ is negative.

This may be proved by the equation

$$\sin(A + 45^\circ) = (\sin A + \cos A) \frac{1}{\sqrt{2}}.$$

51. If the sine of an angle be given, the sine and the cosine of its half are each a four-valued function.

$$\text{For } (\sin \frac{1}{2}\theta + \cos \frac{1}{2}\theta)^2 = 1 + \sin \theta \quad (83),$$

$$(\sin \frac{1}{2}\theta - \cos \frac{1}{2}\theta)^2 = 1 - \sin \theta \quad (83).$$

$$\text{Hence } \sin \frac{1}{2}\theta = \pm \frac{\sqrt{1 + \sin \theta}}{2} \pm \frac{\sqrt{1 - \sin \theta}}{2}, \quad (116)$$

$$\cos \frac{1}{2}\theta = \pm \frac{\sqrt{1 + \sin \theta}}{2} \mp \frac{\sqrt{1 - \sin \theta}}{2}. \quad (117)$$

If the angle θ be given, the preceding lemmas enable us to determine the signs to be given to the radicals in these equations.

52. If the tangent of an angle be given, the tangent of half is two-valued.

Dem.—We have

$$\tan \theta = \frac{2 \tan \frac{1}{2}\theta}{1 - \tan^2 \frac{1}{2}\theta}.$$

Hence, putting $\tan \frac{1}{2}\theta = x$, and $\tan \theta = a$, we have, on determining x , the quadratic equation

$$x^2 + \frac{2}{a}x - 1 = 0, \quad (118)$$

which proves the proposition.

53. If the cosine of an angle be given, the cosine of one-third of the angle is three-valued.

Dem.—In the formula (103) put $a = \frac{\theta}{3}$, and we get

$$\cos \theta = 4 \cos^3 \frac{\theta}{3} - 3 \cos \frac{\theta}{3}.$$

Hence, putting

$$\cos \theta = a, \text{ and } \cos \frac{\theta}{3} = x,$$

$$\text{we get } \frac{3}{4}x^3 - \frac{3}{4}x - \frac{a}{4} = 0, \quad (119)$$

which proves the proposition.

EXERCISES.—XXIX.

Solve the following equations :—

1. $\sin(x + \alpha) - \cos x \cdot \sin \alpha = \cos \alpha.$
2. $\sin(x + \alpha) = b \sin x + c \cos x.$
3. $\cos(x - \alpha) = m \sin x - n \cos x.$
4. $\tan(x + \alpha) + \tan(x - \alpha) = 2 \cot x.$
5. $\sin(\phi + x) = \cos(\phi - x).$
6. $\tan 2x + \tan 3x = 3 \tan x.$
7. $\cos 3x + \sin 3x = (\cos x - \sin x)^3.$
8. $\tan 4x - \tan x = 0.$

9. If $\cos \alpha = \frac{1}{2}$, find the values of $\cos \frac{1}{2}\alpha$ and $\sin \frac{1}{2}\alpha$.

10. Given $\tan 2\theta = \frac{24}{7}$; find the values of $\sin \theta$, $\cos \theta$.

11. Prove the following system of values :—

$$1^{\circ}. \sin \frac{\pi}{4} = \frac{1}{2}\sqrt{2}, \quad \cos \frac{\pi}{4} = \frac{1}{2}\sqrt{2}.$$

$$2^{\circ}. \sin \frac{\pi}{8} = \frac{1}{2}\sqrt{2 - \sqrt{2}}, \quad \cos \frac{\pi}{8} = \frac{1}{2}\sqrt{2 + \sqrt{2}}.$$

$$3^{\circ}. \sin \frac{\pi}{16} = \frac{1}{2}\sqrt{2 - \sqrt{2 + \sqrt{2}}}, \quad \cos \frac{\pi}{16} = \frac{1}{2}\sqrt{2 + \sqrt{2 + \sqrt{2}}}.$$

$$12. \text{Prove } \tan \frac{\pi}{8} = \sqrt{2} - 1.$$

$$13. \text{If } \sin \alpha = \frac{1}{2}, \text{ prove } \tan \frac{1}{2}\alpha = 2 \pm \sqrt{3}.$$

14. If $2 \cos \alpha = -\sqrt{1 + \sin 2\alpha} - \sqrt{1 - \sin 2\alpha}$, between what limits does α lie ?

$$15. \text{If } \sec(\theta + \alpha) + \sec(\theta - \alpha) = 2 \sec \theta,$$

$$\text{prove } \cos \theta = \sqrt{2} \cdot \cos \frac{\alpha}{2}.$$

$$16. \text{Prove } \tan^2(\alpha + 45^\circ) = \frac{\sec 2\alpha + \tan 2\alpha}{\sec 2\alpha - \tan 2\alpha}.$$

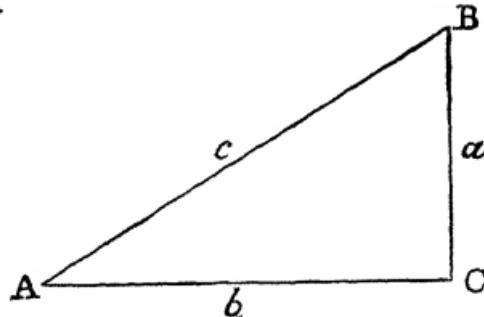
CHAPTER VI.

RELATIONS BETWEEN THE SIDES OF A PLANE TRIANGLE AND THE CIRCULAR FUNCTIONS OF ITS ANGLES.

SECTION I.—*The Right-angled Triangle.*

54. *In a right-angled triangle, any side is equal to the rectangle contained by the hypotenuse, and the sine of the opposite, or the cosine of the adjacent angles.*

Dem.—Let ABC be a triangle having the angle C right; then (Chap. II. Art. 10),



$$BC = AB \sin A, \quad AC = AB \cos A.$$

Hence, denoting the numerical lengths of the sides BC , CA , AB of the triangle by the letters α , b , c , respectively, we have

$$\alpha = c \sin A. \quad (120)$$

$$b = c \cos A. \quad (121)$$

55. *In a right-angled triangle, either side divided by the other side is equal to the tangent of the opposite angle.*

Dem.—Dividing equation (120) by (121), we get

$$\frac{\alpha}{b} = \tan A, \quad \text{or} \quad \alpha = b \tan A. \quad (122)$$

$$\text{Cor.}—\alpha = c \cos B, \quad b = c \sin B, \quad a = b \cot B. \quad (123)$$

Observation.—The three equations (120)–(122) are sufficient for the solution of all the cases of the right-angled triangle.

EXERCISES.—XXX.

1. Prove $\tan^2(45 - \frac{1}{2}A) = \frac{c-a}{c+a} = \tan^2 \frac{1}{2}B.$

2. " $\tan^2(45 + \frac{1}{2}A) = \frac{c+a}{c-a} = \cot^2 \frac{1}{2}B.$

3. " $\sin 2A = \frac{2ab}{c^2}.$

4. " $\cos 2A = \frac{b^2 - a^2}{c^2}.$

5. " $\tan 2A = \frac{2ab}{b^2 - a^2}.$

6. " $\sin 3A = \frac{3ab^2 - a^3}{c^3}.$

7. " $\cos 3A = \frac{b^3 - 3a^2b}{c^3}.$

8. " $\tan \frac{1}{2}A = \frac{a}{b+c}.$

9. " $\sin^2 \frac{1}{2}A = \frac{c-b}{2c}.$

10. " $\cos^2 \frac{1}{2}A = \frac{c+b}{2c}.$

11. In a plane triangle, the altitude divides the base into segments proportional to the cotangents of the adjacent base angles.

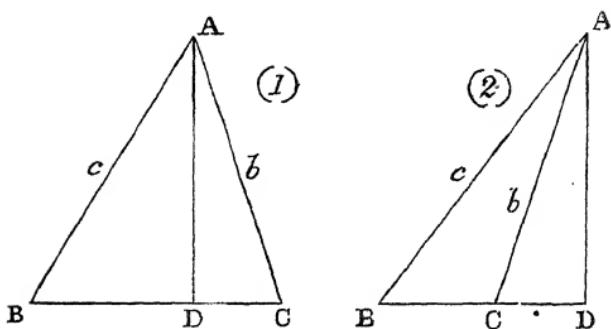
12. In a plane triangle, the altitude divides the vertical angle into two segments whose cosines are reciprocally proportional to the adjacent sides.

13. A tower and its spire subtend equal angles at a point whose distance from the foot of the tower is a ; if h be the height of the tower, prove that the height of the spire is

$$\left(\frac{a^2 + h^2}{a^2 - h^2}\right)h.$$

SECTION II.—*Oblique-angled Triangles.*

56. In any plane triangle, the sides are proportional to the sines of the opposite angles.



This proposition has been already proved in Chap. IV. Art. 31, *Cor.* The following is the proof usually given:—Let ABC be any triangle; from A draw AD perpendicular to the opposite side.

1°. Let B , C be acute angles; then, from fig. (1), we have

$$AD = AB \sin B, \quad AD = AC \sin C;$$

therefore $AC \sin C = AB \sin B.$

Hence $AC : AB :: \sin B : \sin C;$

or $b : c :: \sin B : \sin C.$

2°. If the angle C be obtuse, we have, from fig. (2),

$$AD = AC \sin ACD = AC \sin ACB;$$

since supplemental angles have eqⁿ
before,

$$AD = AB \sin B.$$

Hence $AC \sin C = AB \sin$
therefore $b : c :: \sin B : \sin$

3°. If the angle C be right, we h

$$a : c :: \sin A : 1;$$

but $1 = \sin 90^\circ = \sin C$
therefore $a : c :: \sin A : \sin C$

Hence, in every case the sides are
the sines of the opposite angles, or,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

57. *The sum of any two sides of a triangle divided by the third side as the cosine of half difference of the contained angles is to the sine of half the contained angle.*

Dem.—We have $\frac{a}{c} = \frac{\sin A}{\sin C}$, for

and $\frac{b}{c} = \frac{\sin B}{\sin C}$;

therefore

$$\frac{a+b}{c} = \frac{\sin A + \sin B}{\sin C} = \frac{2 \sin \frac{1}{2}(A+B)}{2 \sin \frac{1}{2}C}$$

but since $\frac{1}{2}(A+B)$ is the complement of

$$\sin \frac{1}{2}(A+B) = \cos \frac{1}{2}C.$$

Hence

$$\frac{a+b}{c} = \frac{\cos \frac{1}{2}(A-B)}{\sin \frac{1}{2}C}.$$

58. *The difference of any two sides of a triangle is to the third side as the sine of half the difference of the opposite angles is to the cosine of half the contained angle.*

Dem.—We have, from (124),

$$\frac{z - b}{c} = \frac{\sin A - \sin B}{\sin C} = \frac{z \cos \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)}{z \sin \frac{1}{2}C \cos \frac{1}{2}C};$$

but $\cos \frac{1}{2}(A + B) = \sin \frac{1}{2}C;$

therefore $\frac{a - b}{c} = \frac{\sin \frac{1}{2}(A - B)}{\cos \frac{1}{2}C}. \quad (126)$

59. *The sum of any two sides of a triangle is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference.*

Dem.—From (124), we have

$$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)} \quad (77). \quad (127)$$

We get the same result if we divide (125) by (126).

EXERCISES.—XXXI.

1. If AD bisects the angle A of the triangle ABC ; prove $BD : DC :: \sin C : \sin B$.

2. If AD' bisect the external vertical angle; prove $BD' : CD' :: \sin C : \sin B$.

3. Hence prove $\frac{1}{DC} = \frac{z \cos \frac{1}{2}A \cdot \cos \frac{1}{2}(B-C)}{a \sin B}$.

4. " " " $\frac{1}{CD'} = \frac{z \sin \frac{1}{2}A \cdot \sin \frac{1}{2}(C-B)}{a \sin B}$.

5. The vertical angle of any triangle is divided by the median that bisects the base into segments, whose sines are inversely proportional to the adjacent sides.

6. If AD be the median that bisects BC ; prove

$$\tan ADB = \frac{2bc \sin A}{b^2 - c^2}.$$

7. In the same case, prove

$$\cot BAD + \cot CAD = 4 \cot A + \cot B + \cot C.$$

60. *In every plane triangle, each side is equal to the sum of the products of the other sides into the cosines of the angles which they make with the first side.*

Dem.—From fig. (1), Art. 56, we have

$$BC = BD + DC = AB \cos B + AC \cos C,$$

that is, $a = c \cos B + b \cos C$.

Again, from fig. (2), we have

$$\begin{aligned} BC &= BD - CD = AB \cos B - AC \cos (180 - C) \\ &= AB \cos B + AC \cos C, \text{ the same as before.} \end{aligned}$$

$$\text{Thus, } a = b \cos C + c \cos B; \quad (128)$$

$$b = c \cos A + a \cos C; \quad (129)$$

$$c = a \cos B + b \cos A. \quad (130)$$

EXERCISES.—XXXII.

1. Prove $a(b \cos C - c \cos B) = b^2 - c^2$.
2. " $(b + c) \cos A + (c + a) \cos B + (a + b) \cos C = a + b + c$.
3. " $a(\cos B \cos C + \cos A) = b(\cos C \cos A + \cos B)$
 $= c(\cos A \cos B + \cos C)$.
4. " $2(a \cos A + b \cos B + c \cos C) = a \cos(B - C)$
 $+ b \cos(C - A) + c \cos(A - B)$.

5. Prove $(a + b + c)(\cos A + \cos B + \cos C) = 2(a \cos^2 \frac{1}{2} A + b \cos^2 \frac{1}{2} B + c \cos^2 \frac{1}{2} C)$.
6. " $c(\cos A + \cos B) = 2(a + b)(\sin^2 \frac{1}{2} C)$.
7. " $c(\cos A - \cos B) = 2(b - a)(\cos^2 \frac{1}{2} C)$.
8. " $\tan B : \tan C = (a^2 + b^2 - c^2) : (a^2 - b^2 + c^2)$.
9. " $a^2 + b^2 + c^2 = 2(ab \cos C + bc \cos A + ca \cos B)$.

61. In any plane triangle, the excess of the sum of the squares of any two sides over the square of the third side is equal to twice their product into the cosine of the contained angle.

Dem.—Multiply the equations (128), (129), (130) by a, b, c respectively, and subtract the first product from the sum of the second and third, and we get

$$b^2 + c^2 - a^2 = 2bc \cos A. \quad (131)$$

And, interchanging letters,

$$c^2 + a^2 - b^2 = 2ca \cos B; \quad (132)$$

$$\text{and } a^2 + b^2 - c^2 = 2ab \cos C. \quad (133)$$

Conversely, equations (128) – (130) can be inferred from equations (131), (133). For, adding the two last and dividing by $2a$, we get (128).

Again, we may prove the proposition that '*the sides are proportional to the sines of the opposite angles*' by the equations of this Article.

For we get, from (131),

$$4b^2c^2 \cos^2 A = (b^2 + c^2 - a^2)^2;$$

that is,

$$4b^2c^2 - 4b^2c^2 \sin^2 A = 2b^2c^2 - 2c^2a^2 - 2a^2b^2 + b^4 + c^4 + a^4.$$

Hence

$$\frac{\sin^2 A}{a^2} = \frac{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}{4a^2b^2c^2}; \quad (134)$$

and since the second side of this equation is unaltered by interchange of letters, we see that

$$\frac{\sin^2 A}{a^2} = \frac{\sin^2 B}{b^2} = \frac{\sin^2 C}{c^2},$$

which is the same as (124).

EXERCISES.—XXXIII.

1. If one angle of a triangle be 120° ; prove that the square on the opposite side exceeds the sum of the squares on the remaining sides by their product.
2. If one angle of a triangle be 60° ; prove that the square on the opposite side is less than the sum of the squares on the remaining sides by their product.
3. If ϕ be an auxiliary* angle such that

$$\tan^2 \phi = \frac{4ab \sin^2 \frac{1}{2} C}{(a+b)^2};$$

prove $c = (a+b) \sec \phi$.

4. If ψ be an auxiliary angle, such that

$$\sin^2 \psi = \frac{4ab \cos^2 \frac{1}{2} C}{(a+b)^2};$$

prove $c = (a+b) \cos \psi$.

5. If θ be an auxiliary angle, such that $\cos \theta = \frac{c}{b}$ (if c be less than b); prove

$$\tan^2 \frac{\theta}{2} = \tan \frac{1}{2} (B-C) \tan \frac{1}{2} A.$$

* An auxiliary angle in Trigonometry is one introduced for the purpose of rendering a formula adapted to Logarithmic computation.

6. If θ be the included angle of two adjacent sides a, b of a parallelogram; prove that the squares of its diagonals are

$$a^2 + b^2 + 2ab \cos \theta, \quad a^2 + b^2 - 2ab \cos \theta.$$

7. If d, d' be the diagonals of a quadrilateral, θ their included angle; prove

$$\text{area} = \frac{1}{2} dd' \sin \theta.$$

8. In any triangle, prove

$$c^2 = (a+b)^2 \sin^2 \frac{1}{2} C + (a-b)^2 \cos^2 \frac{1}{2} C.$$

9. In any triangle, prove

$$c^2 = \frac{(a+b)^2 \sin^2 \frac{1}{2} C + (a-b)^2 \cos^2 \frac{1}{2} C}{\cos(A-B)}.$$

10. If A', B', C' be the supplements of the angles A, B, C of a triangle; prove

$$2bc \operatorname{versin} A' + 2ca \operatorname{versin} B' + 2ab \operatorname{versin} C' = (a+b+c)^2.$$

11. Prove $\sin(A-B) : \sin(A+B) :: a^2 - b^2 : c^2$.

62. To express the sine, the cosine, and the tangent, of half an angle of a triangle in terms of the sides.

1°. We have, from (131),

$$2bc \cos A = b^2 + c^2 - a^2, \quad (\alpha)$$

and

$$2bc = 2bc. \quad (\beta)$$

Hence, by subtraction,

$$2bc(1 - \cos A) = a^2 - (b-c)^2;$$

therefore $4bc \sin^2 \frac{1}{2} A = (a+b-c)(a-b+c)$;

or $\sin^2 \frac{1}{2} A = \frac{(a+b-c)(a-b+c)}{4bc}$.

Let $a + b + c = 2s$, so that s is the semiperimeter of the triangle; then

$$a - b + c = 2(s - b), \text{ and } a + b - c = 2(s - c).$$

Hence $\sin \frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}.$ (135)

Similarly $\sin \frac{1}{2}B = \sqrt{\frac{(s-c)(s-a)}{ca}},$ (136)

and $\sin \frac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{ab}}.$ (137)

2°. By adding equations (α) and (β) we get

$$2bc(1 + \cos A) = (b+c)^2 - a^2,$$

or $4bc \cos^2 \frac{1}{2}A = (b+c+a)(b+c-a) = 2s \cdot 2(s-a);$

therefore $\cos \frac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}.$ (138)

Similarly, $\cos \frac{1}{2}B = \sqrt{\frac{s(s-b)}{ca}},$ (139)

and $\cos \frac{1}{2}C = \sqrt{\frac{s(s-c)}{ab}}.$ (140)

3°. By dividing equations (135)-(137) by (138)-(140) respectively, we get

$$\tan \frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}. \quad (141)$$

$$\tan \frac{1}{2}B = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}. \quad (142)$$

$$\tan \frac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}. \quad (143)$$

$$Cor. 1. - \sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}. \quad (144)$$

For

$$\sin A = 2 \sin \frac{1}{2}A \cos \frac{1}{2}A = \frac{2}{bc} \sqrt{s \cdot s-a \cdot s-b \cdot s-c}.$$

EXERCISES.—XXXIV.

1. Prove $\tan \frac{1}{2}A \cdot \tan \frac{1}{2}B = \frac{s-c}{s}$.

2. , , $\tan \frac{1}{2}A \div \tan \frac{1}{2}B = (s-b) \div (s-a)$.

3. , , $\cos^2 \frac{1}{2}A \div \cos^2 \frac{1}{2}B = a(s-a) \div b(s-b)$.

4. , , $\frac{\tan \frac{1}{2}A - \tan \frac{1}{2}B}{\tan \frac{1}{2}A + \tan \frac{1}{2}B} = \frac{a-b}{c}$.

5. , , $2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4$
 $= 16s \cdot s-a \cdot s-b \cdot s-c$.

6. , , $(b-c) \cot \frac{A}{2} + (c-a) \cot \frac{B}{2} + (a-b) \cot \frac{C}{2} = 0$.

7. , , $bc \cos^2 \frac{A}{2} + ca \cos^2 \frac{B}{2} + ab \cos^2 \frac{C}{2} = s^2$.

8. , , $b \cos^2 \frac{C}{2} + c \cos^2 \frac{B}{2} = s$.

9. , , $b \sin^2 \frac{C}{2} + c \sin^2 \frac{B}{2} = s-a$.

10. Prove $(s-a) \tan \frac{A}{2} = (s-b) \tan \frac{B}{2} = (s-c) \tan \frac{C}{2}$.

11. $\therefore a^2 \sin 2B + b^2 \sin 2A = 2ab \sin C$.

12. $\therefore s^2 \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2} = \sqrt{s \cdot (s-a)(s-b)(s-c)}$.

13. Prove that the length of the line bisecting the angle A of a triangle, and meeting the opposite side, is

$$\frac{2bc \cos \frac{A}{2}}{b+c}.$$

14. If x, y, z be three angles determined by the relations

$$\cos x = \frac{a}{b+c}, \quad \cos y = \frac{b}{c+a}, \quad \cos z = \frac{c}{a+b};$$

prove $\tan^2 \frac{x}{2} + \tan^2 \frac{y}{2} + \tan^2 \frac{z}{2} = 1$,

$$\therefore \tan \frac{x}{2} \cdot \tan \frac{y}{2} \cdot \tan \frac{z}{2} = \tan \frac{A}{2} \cdot \tan \frac{B}{2} \cdot \tan \frac{C}{2}.$$

15. If in a triangle $\sin C = \frac{\sin A + \sin B}{\cos A + \cos B}$; prove $C = \frac{\pi}{2}$.

16. Prove $\frac{\tan \frac{1}{2}A - \tan \frac{1}{2}B}{\tan \frac{1}{2}A + \tan \frac{1}{2}B} = \frac{a+b}{c}$.

17. $\therefore \frac{\text{vers } A}{\text{vers } B} = \frac{a(s-b)}{b(s-a)}$, and $\frac{\text{vers}(A+B)}{\text{vers } C} = \frac{s(s-c)}{(s-a)(s-b)}$.

CHAPTER VII.

PROPERTIES OF TRIANGLES, ETC.

63. To find Expressions for the Area of a Triangle.

1°. *The area of a triangle is equal to half the product of any two sides into the sine of their included angle.*

Dem.—Let ABC be the triangle, CD the perpendicular from C on AB ; then
(Euc. II. 1. Cor. 2) the area of

$$ABC = \frac{1}{2} AB \cdot CD;$$

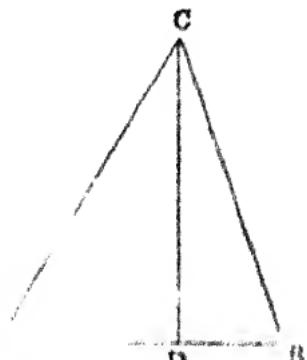
$$\text{but } CD = AC \sin A.$$

Hence

$$\text{area} = \frac{1}{2} AB \cdot AC \sin A;$$

that is,

$$\text{area} = \frac{1}{2} bc \sin A. \quad (145)$$



2° *The area of a triangle in terms of its sides.*

Since $\sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}$ (144);

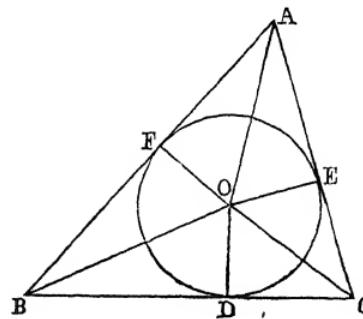
substituting this in (145), we get

$$\text{area of the triangle} = \sqrt{s(s-a)(s-b)(s-c)}. \quad (146)$$

DEF.—The area of the triangle shall be denoted by S .

64. To find the Radius of the Inscribed Circle of a Triangle.

Let ABC be the triangle; r the radius of the circle; O the centre; D, E, F the points of contact.



$$\text{Then area of triangle } BOC = \frac{ar}{2};$$

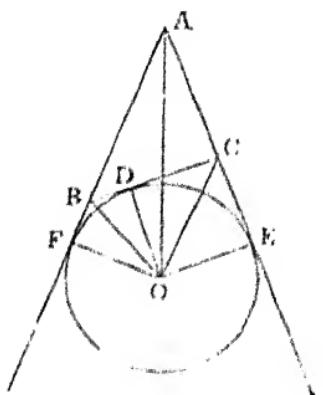
$$\text{, , } COA = \frac{br}{2};$$

$$\text{, , } AOB = \frac{cr}{2};$$

$$\text{therefore } S = \left(\frac{a+b+c}{2} \right) r = rs;$$

$$\text{therefore } r = \frac{S}{s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \quad (147)$$

65. To find the Radii of the Escribed Circles of a Triangle.



Denoting the radii of the escribed circles by r' , r'' , r''' , we have the

$$\text{area of triangle } BOC = \frac{1}{2} ar';$$

$$\text{,, } COA = \frac{1}{2} br';$$

$$\text{,, } AOB = \frac{1}{2} cr'.$$

Hence, adding the two last, and subtracting the first, we get

$$S = \frac{1}{2} (b + c - a) r' = (s - a) r';$$

therefore $r' = \frac{S}{s - a}.$ (148)

Similarly, $r'' = \frac{S}{s - b},$ (149)

and $r''' = \frac{S}{s - c}.$ (150)

Alternative proofs of the equations (47)–(50) may be given. Thus, see fig. (147), we have

$$OF \div AF = \tan \frac{1}{2} A;$$

that is (*Sequel*, IV. Prop. 1.),

$$r \div (s - a) = \tan \frac{1}{2} A.$$

Hence $r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \frac{s}{s};$

and similarly for the others.

66. To find the Circumradius of a Triangle in Terms of its Sides.

Let R be the circumradius; then (Art. 31)

$$\frac{a}{2R} = \sin A = \frac{2S}{bc} \quad (145);$$

therefore

$$\frac{a}{2R} = \frac{2S}{bc}.$$

Hence

$$R = \frac{abc}{4S}. \quad (151)$$

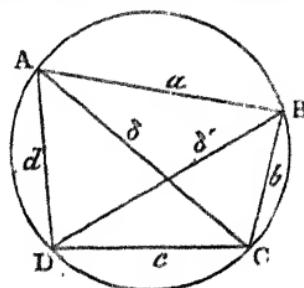
67. To find the Area of a Cyclic Quadrilateral in Terms of its Sides.

Let $ABCD$ be the quadrilateral, and let the sides and diagonals be denoted as in the diagram; then, since the angles B, D are supplemental, we have

$$\delta^2 = a^2 + b^2 - 2ab \cos B$$

(Art. 61) (a),

$$\delta^2 = c^2 + d^2 - 2cd \cos B;$$



and eliminating δ , we get

$$\cos B = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}.$$

Hence $\sin^2 \frac{1}{2}B = \frac{(c + d)^2 - (a - b)^2}{4(ab + cd)},$

and $\cos^2 \frac{1}{2}B = \frac{(a + b)^2 - (c - d)^2}{4(ab + cd)}.$

Now, putting s for the semiperimeter of the quadrilateral, these equations give us

$$\sin \frac{1}{2}B = \sqrt{\frac{(s - a)(s - b)}{ab + cd}}; \quad (152)$$

$$\cos \frac{1}{2}B = \sqrt{\frac{(s - c)(s - d)}{ab + cd}}. \quad (153)$$

Hence

$$\sin B = \frac{2\sqrt{(s - a)(s - b)(s - c)(s - d)}}{ab + cd}. \quad (154)$$

Now, if S denote the area of the quadrilateral, we have

$$S = \frac{1}{2}(ab + cd) \sin B \text{ (Art. 63);}$$

therefore $S = \sqrt{(s - a)(s - b)(s - c)(s - d)}. \quad (155)$

Cor. 1—If we substitute the value of $\cos B$ in equation (a), page 81, we get

$$\delta^2 = \frac{(ac + bd)(ad + bc)}{(ab + cd)}. \quad (156)$$

Similarly, $\delta'^2 = \frac{(ac + bd)(ab + cd)}{(ad + bc)}. \quad (157)$

Cor. 2.—If R be the circumradius, we have

$$R = \frac{\delta}{2 \sin B}.$$

Hence $4R = \sqrt{\frac{(ac + bd)(ab + cd)(ad + bc)}{(s - a)(s - b)(s - c)(s - d)}}.$ (158)

68. *To express the area of any quadrilateral in terms of its sides and a pair of opposite angles.*

We have (see last diagram),

$$S = \frac{1}{2}(ad \sin A + bc \sin C);$$

but $\cos A = \frac{a^2 + d^2 - \delta'^2}{2ad}, \quad \cos C = \frac{b^2 + c^2 - \delta'^2}{2bc}.$

Hence

$$2ad \cos A - 2bc \cos C = a^2 + d^2 - b^2 - c^2;$$

therefore

$$(a + d)^2 - (b - c)^2 = 4ad \cos^2 \frac{1}{2}A + 4bc \sin^2 \frac{1}{2}C,$$

and

$$(b + c)^2 - (a - d)^2 = 4ad \sin^2 \frac{1}{2}A + 4bc \cos^2 \frac{1}{2}C.$$

Hence, multiplying and reducing, we get

$$(s - a)(s - b)(s - c)(s - d) = \frac{1}{4}(ad \sin A + bc \sin C)^2 + abcd \cos^2 \frac{1}{2}(A + C);$$

therefore

$$S^2 = (s - a)(s - b)(s - c)(s - d) - abcd \cos^2 \frac{1}{2}(A + C).$$

59. Being given the side of a regular polygon of n sides, to find the radii of its inscribed and circumscribed circles.

Let AB , a side of the polygon, be denoted by a . Let O be the common centre of the circles; R, r their radii. Let fall the perpendicular OD ; then

the angle $ADO = \frac{\pi}{2}$; and

in the triangle AOD we have

$$AO \cdot \sin AOD = AD;$$

that is,

$$R \cdot \sin \frac{\pi}{n} = \frac{a}{2};$$

therefore

$$R = \frac{a}{2 \sin \frac{\pi}{n}}. \quad (160)$$

Again,

$$AD \div OD = \tan AOD;$$

that is,

$$\frac{a}{2} \div r = \tan \frac{\pi}{n};$$

therefore

$$r = \frac{a}{2 \tan \frac{\pi}{n}}. \quad (161)$$

Cor. 1.—The area of the polygon, being n times the triangle AOB , is

$$= n \cdot \frac{a}{2} \cdot r = na^2 \cdot \cot \frac{\pi}{n}. \quad (162)$$

$$\text{Cor. 2.—Area} = nR^2 \cdot \sin \frac{\pi}{n} \cos \frac{\pi}{n}. \quad (163)$$

$$\text{Cor. 3.—Area} = nr^2 \cdot \tan \frac{\pi}{n}. \quad (164)$$

Cor. 4.—The area of a circle, whose radius is r , is πr^2 . (165)

For the circle may be regarded as the limit of an inscribed polygon of an indefinitely large number of sides. Now if n be indefinitely large,

$$\tan \frac{\pi}{n} = \frac{\pi}{n}.$$

Hence, from *Cor. 3*,

$$\text{area} = nr^2 \cdot \frac{\pi}{n} = \pi r^2.$$

Cor. 5.—If θ be the circular measure of the angle of a sector,

$$\text{area of sector} = \frac{\theta r^2}{2}. \quad (166)$$

EXERCISES.—XXXV.

1. The sides of a triangle are 18, 24, 30; find the radii of its inscribed and escribed circles, and the diameter of its circumscribed circle.

2. If two angles of a triangle be 75° and 45° , respectively, and the included side 24 feet, find its area. Prove

$$\text{area} = \frac{1}{2} \frac{c^2}{\cot A + \cot B}.$$

3. Prove that the bisectors of the angles A , B , C of a triangle are, respectively, equal to

$$\frac{2bc \cos \frac{1}{2}A}{b+c}, \quad \frac{2ca \cos \frac{1}{2}B}{c+a}, \quad \frac{2ab \cos \frac{1}{2}C}{a+b}.$$

4. Prove that the lengths of the sides of the pedal triangle, that is the triangle formed by joining the feet of the perpendiculars, are—

$$a \cos A, b \cos B, c \cos C, \text{ respectively.}$$

5. Prove $\frac{r}{r'} = \frac{1}{r'} + \frac{1}{r''} + \frac{1}{r'''}$

6. " $r' r'' r''' = r^3 \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$.

7. " $r' = \frac{a \sin \frac{1}{2} B \sin \frac{1}{2} C}{\cos \frac{1}{2} A}$.

8. " $r' = \frac{a \cos \frac{1}{2} B \cos \frac{1}{2} C}{\cos \frac{1}{2} A}$.

9. Prove that the angles of the pedal triangle are, respectively,

$$\pi - 2A, \quad \pi - 2B, \quad \pi - 2C.$$

10. Prove the area $S = \frac{a^2 + b^2}{2(\cot B + \cot A)}$.

11. " " " = $\sqrt{r r' r'' r'''}$.

12. " " " = $\frac{r' r'' r'''}{S}$.

13. " " " = $\frac{abc}{a+b+c} \left(\cos \frac{A}{2}, \cos \frac{B}{2}, \cos \frac{C}{2} \right)$.

14. " " " = $2R^2 \sin A, \sin B, \sin C$.

15. " " " = $4Rr (\cos \frac{1}{2} A, \cos \frac{1}{2} B, \cos \frac{1}{2} C)$.

16. " " " = $r^2 \cot \frac{1}{2} A, \cot \frac{1}{2} B, \cot \frac{1}{2} C$.

17. Prove that the area of the in-circle : area of triangle :: $\pi : \cot \frac{1}{2} A, \cot \frac{1}{2} B, \cot \frac{1}{2} C$.

18. Prove that the perimeter of the pedal triangle is equal to

$$4R \sin A \sin B \sin C.$$

In the identities, 19-28 inclusive, $A + B + C = \pi$.

19. Prove $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$.

20. , , $\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$.

21. , , $\sin A + \sin B - \sin C = 4 \sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}$.

22. , , $\cos A + \cos B + \cos C - 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = 1$.

23. , , $\cos 2A + \cos 2B + \cos 2C + 4 \cos A \cos B \cos C + 1 = 0$.

24. , , $\cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2}$
 $= 4 \cos \frac{\pi - A}{4} \cos \frac{\pi - B}{4} \cos \frac{\pi - C}{4}$.

25. , , $\cos \frac{A}{2} + \cos \frac{B}{2} - \cos \frac{C}{2}$
 $= 4 \sin \frac{\pi - A}{4} \sin \frac{\pi - B}{4} \cos \frac{\pi - C}{4}$.

26. , , $\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C - 1 = 0$.

27. , , $2 \sin^2 A \sin^2 B + 2 \sin^2 B \sin^2 C + 2 \sin^2 C \sin^2 A$
 $= \sin^4 A + \sin^4 B + \sin^4 C + 8 \sin A \sin B \sin C$.

28. , , $\sin^3 A + \sin^3 B + \sin^3 C$

$$= 3 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} + \cos \frac{3A}{2} \cos \frac{3B}{2} \cos \frac{3C}{2}$$

29. If a, b, c be the lengths through the vertices and the circumcentre to meet the opposite sides; prove

$$\frac{a}{2R} = \frac{\sin B \sin C}{\cos(B-C)}, \quad \frac{b}{2R} = \frac{\sin C \sin A}{\cos(C-A)}, \quad \frac{c}{2R} = \frac{\sin A \sin B}{\cos(A-B)}$$

30. Prove $2R + 2r = a \cot A + b \cot B + c \cot C$.

31. Prove $a = \frac{r \cos \frac{1}{2}A}{\sin \frac{1}{2}B \sin \frac{1}{2}C} = \frac{r' \cos \frac{1}{2}A}{\cos \frac{1}{2}B \cos \frac{1}{2}C}$.

32. If a' , b' , c' be the sides of the triangle formed by joining the centres of the escribed circles ; prove

$$a = a' \sin \frac{A}{2}, \quad b = b' \sin \frac{B}{2}, \quad c = c' \sin \frac{C}{2}.$$

33. In the same case prove that the area of the triangle whose sides are a' , b' , c' is $\frac{abc}{r}$.

34. Prove that the distances of the centres of the escribed circles from the centre of the inscribed are

$$a \sec \frac{A}{2}, \quad b \sec \frac{B}{2}, \quad c \sec \frac{C}{2},$$

respectively.

35. The perpendiculars of a triangle are respectively equal to

$$\frac{2s}{\cot \frac{1}{2}B + \cot \frac{1}{2}C}, \quad \frac{2s}{\cot \frac{1}{2}C + \cot \frac{1}{2}A}, \quad \frac{2s}{\cot \frac{1}{2}A + \cot \frac{1}{2}B}.$$

36. If e , e' , e'' be the reciprocals of the perpendiculars of a triangle, and if $e + e' + e'' = 2s$; prove

$$\sqrt{\sigma(\sigma - e)(\sigma - e')(\sigma - e'')} = \frac{1}{4S}.$$

37. In the same case prove

$$\sin A = \frac{2\sqrt{\sigma(\sigma - e)(\sigma - e')(\sigma - e'')}}{2e'e''}.$$

38. If $xy + yz + zx = 1$; prove, by Trigonometry, that

$$\frac{x}{1-x^2} + \frac{y}{1-y^2} + \frac{z}{1-z^2} = \frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)^2}.$$

39. If r_a , r_b , r_c denote the radii of the three circles, each touching the in-circle and two sides of a triangle ; prove

$$r_a = r \tan^2 \frac{1}{4}(B+C), \quad r_b = r \tan^2 \frac{1}{4}(C+A), \quad r_c = r \tan^2 \frac{1}{4}(A+B).$$

40. Prove

$$S = \frac{1}{4} \left(\frac{a^2 + b^2 + c^2}{\cot A + \cot B + \cot C} \right) = \frac{1}{4}(a^2 \cot A + b^2 \cot B + c^2 \cot C).$$

41. Prove

$$\frac{(a+b+c)^2}{a^2 + b^2 + c^2} = \frac{\cot \frac{1}{2}A + \cot \frac{1}{2}B + \cot \frac{1}{2}C}{\cot A + \cot B + \cot C}.$$

42. If R, r denote the radii of the circumscribed and inscribed circles to a regular polygon of any number of sides; R', r' corresponding radii to a regular polygon of the same area, and double the number of sides; prove

$$R' = \sqrt{Rr}, \quad \text{and} \quad r' = \sqrt{\frac{r(R+r)}{2}}.$$

43. If A', B', C' be the angles subtended by the sides of a triangle at the centre of its inscribed circle; prove

$$4 \sin A' \sin B' \sin C' = \sin A + \sin B + \sin C.$$

44. If $A + B + C = \pi$; prove

$$\frac{\cos A}{\sin B \sin C} = \frac{\cos B}{\sin C \sin A} + \frac{\cos C}{\sin A \sin B} = 2.$$

45. If the distances of the centre of the inscribed circle from the angles A, B, C of a triangle be denoted by d, e, f , respectively; prove

$$d^2 \left(\frac{1}{b} - \frac{1}{c} \right) + e^2 \left(\frac{1}{c} - \frac{1}{a} \right) + f^2 \left(\frac{1}{a} - \frac{1}{b} \right) = 0;$$

and $\frac{\cos \frac{1}{2}A}{d} + \frac{\cos \frac{1}{2}B}{e} + \frac{\cos \frac{1}{2}C}{f} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

CHAPTER VIII.

SOLUTION OF TRIANGLES

70. Every triangle has six parts, namely, the three sides and the three angles. When any three of these six parts, except the three angles, are given, the three remaining parts can be calculated; the process of doing which is called *the solution of triangles*. The reason that the three angles are insufficient is, that they are not independent; for (Euc. I. xxxii.) if two of them be given, the third is determined. Triangles in trigonometry are divided into right-angled and oblique. The solution of the former has been already given. We now give that of the latter.

71. There are four cases of oblique-angled triangles—

- i. *A side and two angles.*
- ii. *Two sides and an angle opposite to one of them.*
- iii. *Two sides and the included angle.*
- iv. *The three sides.*

Case I.—Suppose B and C are the given angles, and a the given side :

then
$$A = 180^\circ - (B + C). \quad (167)$$

 ii

Again, $\frac{b}{a} = \frac{\sin B}{\sin A};$

therefore $\log b - \log a = \log \sin B - \log \sin A.$

Hence, adding 10 to $\log \sin B$ and $\log \sin A$, we get

$$\log b = \log a + \text{Log sin } B - \text{Log sin } A. \quad (168)$$

Similarly,

$$\log c = \log a + \text{Log sin } C - \text{Log sin } A. \quad (169)$$

The equations (167)–(169) determine the required parts.

If A, B be the given angles, we have

$$C = 180^\circ - (A + B),$$

and b, c are found as before.

EXAMPLE.—Given

$$B = 38^\circ 12' 48'', \quad C = 60^\circ, \quad a = 7012.5,$$

it is required to find the sides b, c .

Type of the Calculation.

$$A + B + C = 180; \therefore A = 81^\circ 47' 12''.$$

$$\log a = 3.8458729, \quad \text{Log sin } C = 9.9375306.$$

$$\text{Log sin } B = 9.7914038, \quad \text{Hence } \log c = 3.7878810;$$

$$\text{Log sin } A = 9.9955225, \quad \therefore c = 6135.94.$$

$$\therefore \log b = 3.6417542.$$

$$\text{Hence } b = 4382.82.$$

EXERCISES.—XXXVI.

1. Given $a = 90$, $B = 50^{\circ} 30'$, $C = 122^{\circ} 45'$; calculate b , c , A .
2. " $a = 479$, $A = 82^{\circ} 22'$, $B = 43^{\circ} 20'$; calculate b , c , C .
3. " $a = 795$, $A = 79^{\circ} 50'$, $B = 41^{\circ} 41'$; calculate b , c , C .
4. " $a = 999$, $B = 37^{\circ} 58'$, $C = 65^{\circ} 2'$; calculate b , c , A .
5. " $a = 6412$, $A = 70^{\circ} 55'$, $C = 52^{\circ} 4'$; calculate b , c , B .
6. A transversal divides one of the angles of an equilateral triangle in the ratio of $2 : 1$; in what ratio does it divide the opposite side?
7. If a parallel XV to the base ZC of a triangle ZBC be drawn, so that $XV = ZX + CY$; find the length of XV , being given
 $a = 2979$, $B = 47^{\circ} 24'$, $C = 97^{\circ} 22'$.
8. Being given the angles of a trapezium and the lengths of the parallel sides; show how to calculate the remaining sides.

72. Case II.—Given a , b , and the angle A , to calculate B , C , c .

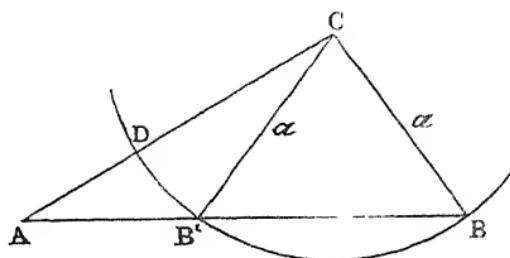
We have $\frac{\sin B}{\sin A} = \frac{b}{a}$.

Hence

$$\log \sin B = \log \sin A + \log b - \log a. \quad (170)$$

Since the sine of an angle is equal to the sine of its supplement, when an angle of a triangle is found

from its sine, it is sometimes doubtful which of the two supplemental angles having the given sine is to be selected. *This happens when two sides and the angle opposite to the less are given.* Thus, if ACB be a triangle having AC greater than CB ; then, if CD be cut off equal to CB , the circle described with C as centre and CD as radius will cut the base AB in two points, B, B' ; then, joining CB', CB , we have two tri-



angles, ACB, ACB' , having the parts b, a, A exactly the same in both, although their remaining parts are different. Hence these two triangles fulfil the required conditions. On this account this is called the *ambiguous case* in the solution of triangles.

When two sides and the angle opposite to the greater are given, there will be no ambiguity, because the angle opposite to the less must be acute.

73. The angle B having been determined, C is given by the equation

$$A + B + C = 180^\circ,$$

and then the third side can be calculated as in Case I. It can also be found as follows:—

We have $b^2 + c^2 - 2bc \cos A = a^2$; (171)

and solving as a quadratic we get

$$c = b \cos A \pm \sqrt{a^2 - b^2 \sin^2 A}.$$

If a be less than $b \sin A$, the values of c will be imaginary, and there will be no triangle answering to the given conditions. If $a = b \sin A$, the angle B will be right, and there will be no ambiguity. Lastly, if a be greater than $b \sin A$, there will be two real values for c . In order that each may be positive, $b^2 \cos^2 A$ must be greater than $a^2 - b^2 \sin^2 A$, or a must be less than b , as before. Hence, in the ambiguous case, a must be less than b and greater than $b \sin A$.

EXAMPLE.—Given $a = 7$, $b = 8$, $A = 27^\circ 47' 45''$, to find B , C , c .

Type of the Calculation.

$$\text{Log } \sin B = \text{Log } \sin A + \log b - \log a.$$

$$\text{Log } \sin A = .6686860.$$

$$\log b = .9030900.$$

$$\log a = .8450980;$$

$$\text{therefore } \text{Log } \sin B = 9.7266780.$$

There are two solutions—

$$B = 32^\circ 12' 15'', \text{ or } B = 147^\circ 47' 45''.$$

$$\text{Hence } C = 120^\circ, \quad \text{or } C = 4^\circ 24' 30'';$$

$$\text{and } c = 13, \quad \text{or } c = 1.15385.$$

EXERCISES.—XXXVII.

1. Given $a = 345$, $b = 695$, $A = 21^\circ 14' 25''$,
to find B, C, c .
2. „ $a = 77.04$, $b = 91.06$, $A = 41^\circ 13' 0''$,
to find B, C, c .
3. „ $a = 309$, $b = 360$, $A = 21^\circ 14' 25''$,
to find B, C, c .
4. „ $a = 83.856$, $b = 83.153$, $B = 68^\circ 10' 24''$,
to find A, C, c .
5. „ $a = 27.548$, $b = 35.055$, $B = 60^\circ 0' 32''$,
to find A, C, c .

6. Given of a parallelogram a side $a = 35$, a diagonal $d = 63$, and the angle between the diagonals $= 21^\circ 36' 30''$; it is required to calculate the remaining side and the remaining diagonal.

74. Case III.—Given a, b and the angle C , to find A, B, c .

From Art. 59 we get

$$a + b : a - b :: \tan \frac{1}{2}(A + B) : \tan \frac{1}{2}(A - B).$$

Hence

$$\begin{aligned} \text{Log } \tan \frac{1}{2}(A - B) &= \text{log } (a - b) + \text{Log } \tan \frac{1}{2}(A + B) \\ &\quad - \text{log } (a + b). \end{aligned} \quad (172)$$

Now, since C is given, its supplement $(A + B)$ is given; and equation (172) determines $\frac{1}{2}(A - B)$. Hence A and B are found.

The side c can be computed from formula, Art. 57, which in logarithms is

$$\text{log } c = \text{log } (a + b) + \text{Log } \sin \frac{1}{2}C - \text{Log } \cos \frac{1}{2}(A - B). \quad (173)$$

n be found also by means of the p , ψ of Exercises 3, 4, Art. 61.

iven $a = 601$, $b = 289$, $C = 100^\circ 19' 6''$,

Type of the Calculation.

$$\log a - b = 2.4941546$$

$$\log a + b = 2.9493900$$

$$'54'' \quad \text{Log} \tan \frac{1}{2}(A+B) = 9.9213621$$

$$\text{Log} \tan \frac{1}{2}(A-B) = 9.4661267$$

$$\therefore \frac{1}{2}(A-B) = 16^\circ 18' 15''$$

$$, 50' 27'';$$

$$\therefore A = 56^\circ 8' 42'',$$

$$B = 23^\circ 32' 12''.$$

EXERCISES.—XXXVIII.

232, $b = 229$, $C = 15^\circ 11' 21''$,
 to find A , B , c .

2332, $b = 3476$, $C = 126^\circ 12' 14''$,
 to find A , B , c .

20.71, $b = 18.87$, $C = 55^\circ 12' 3''$,
 to find A , B , c .

8.54, $b = 6.39$, $C = 12^\circ 35' 8''$,
 to find A , B , c .

3184, $b = 917$, $C = 34^\circ 9' 16''$,
 to find A , B , c .

an equilateral triangle be divided into three
ulate the portions into which the opposite
by the lines connecting it with the points of

the diagonals of a parallelogram and their
show how to calculate the sides.

75. Case IV.—To solve a triangle, being given the three sides.

We have $r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$ (equation (64));

and $\tan \frac{1}{2}A = \frac{r}{s-a}$ (Art. 65).

Hence

$$\log r = \frac{1}{2}\{\log(s-a) + \log(s-b) + \log(s-c) - \log s\}. \quad (174)$$

$$\text{Log } \tan \frac{1}{2}A = 10 + \log r - \log(s-a). \quad (175)$$

Similarly,

$$\text{Log } \tan \frac{1}{2}B = 10 + \log r - \log(s-b), \quad (176)$$

$$\text{and } \text{Log } \tan \frac{1}{2}C = 10 + \log r - \log(s-c). \quad (177)$$

EXAMPLE.—Given

$$a = 13, \quad b = 14, \quad c = 15; \quad \text{calculate } A, B, C.$$

Type of the Calculation.

$a = 13$	$\log(s-a) = .9030900$
$b = 14$	$\log(s-b) = .8450980$
$c = 15$	$\log(s-c) = .7781513$
<hr/>	<hr/>
$2s = 42$	2.5263393
<hr/>	<hr/>
$s = 21$	$\log s = 1.3222193$
$s-a = 8$	<hr/>
$s-b = 7$	1.2041200
$s-c = 6$	$\log r = .6020600$
	$\therefore 10 + \log r = 10.6020600$

Hence $\log \tan \frac{1}{2} A = 9.6989700.$

,, $\log \tan \frac{1}{2} B = 9.7569620.$

,, $\log \tan \frac{1}{2} C = 9.8239087.$

Hence

$$A = 53^\circ 7' 48.38'', \quad B = 59^\circ 29' 23.18'',$$

$$C = 67^\circ 22' 48.44''.$$

EXERCISES.—XXXIX.

1. Given $a = 317, \quad b = 533, \quad c = 510;$ calculate $A, B, C.$
2. ,,, $a = 289, \quad b = 601, \quad c = 712;$,,, $A, B, C.$
3. ,,, $a = 17, \quad b = 113, \quad c = 120;$,,, $A, B, C.$
4. ,,, $a = 15.47, \quad b = 17.39, \quad c = 22.88;$,,, $A, B, C.$
5. ,,, $a = 5134, \quad b = 7268, \quad c = 9313;$,,, $A, B, C.$
6. ,,, $a = 99, \quad b = 101, \quad c = 158;$,,, $A, B, C.$

MISCELLANEOUS EXERCISES.—XL.

1. Given $a = 7, \quad b = 8, \quad C = 120^\circ;$ find $A, B, C.$
2. ,,, $a = 516, \quad b = 219, \quad C = 98^\circ 54';$,,, $A, B, C.$
3. ,,, $A = 18^\circ, \quad a = 3, \quad b = 3 + \sqrt{45};$ solve the triangle.
4. ,,, $A = 15^\circ, \quad a = 5, \quad b = 5 + \sqrt{75}.$,,,
5. If the angles of a triangle be in the ratio $1 : 2 : 7;$ prove that the greatest side : least :: $\sqrt{5} + 1 : \sqrt{5} - 1.$

6. Prove $a = (s \cdot \sin \frac{1}{2} A) \div \cos \frac{1}{2} B \cos \frac{1}{2} C$.

7. The angles of a triangle are as $1 : 2 : 3$, and the difference between the greatest and the least side is 80 yards; find the area.

8. Given $a = 18$, $b = 2$, $C = 55^\circ$; find A , B , being given $\log 2 = .3010300$, $\log \tan 62^\circ 30' = 10.2835233$, $\log \tan 56^\circ 56' = 10.1863769$, tab. diff. for $1' = 2763$.

9. Given $A = 30^\circ$, $a = 3$, $b = 3\sqrt{3}$; prove $C = 90^\circ$.

10. If C, C' be the two values of the third angle in the ambiguous case, when a, b, A are given, and b greater than a ; prove $\tan A = \cot \frac{1}{2}(C + C')$.

11. The area of an in-polygon : area of corresponding circum-polygon as $3 : 4$; find the number of sides.

12. Given $\log \sin 59^\circ 37' 40'' = 9.9358894$, diff. for $10'' = 124$, and $\log \sin A = 9.9358921$; find A .

13. The sides of a triangle are as $9 : 7$, and included angle $= 64^\circ 12'$; determine the remaining angles, being given $\log 2 = .3010300$, $\log \tan 57^\circ 54' = 10.2025255$, $\log \tan 11^\circ 16' = 9.2993216$, $\log \tan 11^\circ 17' = 9.2998804$.

14. If the alternate angles of a regular pentagon be joined; prove interior : original pentagon $:: 3 - \sqrt{5} : 3 + \sqrt{5}$.

Solve the questions 15-18, without logarithms.

15. $b = 3$, $c = 2\sqrt{3}$, $A = 30^\circ$; prove $C = 90^\circ$.

16. $a = 2\sqrt{3}$, $b = 3 - \sqrt{3}$, $c = 3\sqrt{2}$; $\therefore C = 120^\circ$.

17. $a = 2$, $b = 1 + \sqrt{3}$, $c = \sqrt{6}$; $\therefore C = 60^\circ$.

18. $a = 12$, $b = \frac{399}{40}$, $A = 45^\circ$; $\therefore B = 36^\circ$.

19. If the angle A of a triangle be very obtuse, show that the circular measure of the sum of B and C is very nearly

$$\sqrt{\frac{(a+b+c)(b+c-a)}{bc}}.$$

20. If the side of the base of a square pyramid :: length of an edge :: 4 : 3; find the slope of each face, being given $\log 2 = .3010300$, $\log \tan 26^{\circ} 33' = .96986800$, diff. for 1' = 3200.

21. The sides of a triangle are 7, 8, 9; find the areas of the inscribed and escribed circles.

22. If an angle of a triangle be 60°, and the including sides 19 and 1; find the other angles, being given $\log 3 = .4771213$, $\log \tan 57^{\circ} 19' 11'' = 10.1928032$.

23. Prove $c^2 : a^2 - b^2 :: \sin C : \sin (A - B)$.

24. The three sides of a triangle are 9, 10, 11; find $\log \tan$ of the smallest angle, being given $\log 2 = .3010300$, $\log 3 = .4771213$.

25. If $\tan \frac{1}{2} A$, $\tan \frac{1}{2} B$, $\tan \frac{1}{2} C$ be in GP, prove

$$b < \frac{a^2 + c^2}{a + c}.$$

26. If in a triangle $A = 2B + 4C$, prove $a = 17 \cdot \sin \frac{\pi}{14}$.

27. Solve a triangle, being given the side a , the angle A , and the area.

28. Solve a triangle, knowing an angle and two altitudes.

29. Solve a triangle; being given the perimeter, an angle, and the area.

30. Solve a triangle; being given the area, the side c , and the difference of the adjacent angles.

CHAPTER IX.

APPLICATION OF THE SOLUTION OF TRIANGLES TO THE MEASUREMENT OF HEIGHTS AND DISTANCES.

76. To find the Height and Distance of an Inaccessible Object on a Horizontal Plane.

Let CD be the inaccessible object; A, B two points in the plane accessible to each other; and from each of which the summit C of CD can be observed.

i°. Suppose the points A, B, D to be collinear, and let the angles of elevation DAC, DBC be denoted by α, β respectively; then in the triangle ABC , the angle

$$ACB = \alpha - \beta.$$

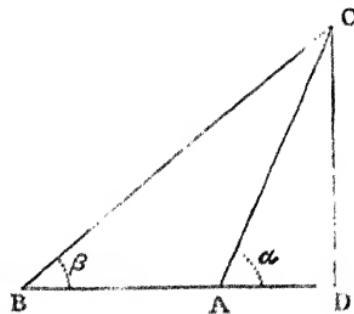
Hence $AC : AB :: \sin \beta : \sin (\alpha - \beta)$;

therefore $AC = \frac{AB \sin \beta}{\sin (\alpha - \beta)}$.

But in the right-angled triangle ADC , we have

$$CD = AC \sin \alpha, \quad AD = AC \cos \alpha;$$

therefore $CD = \frac{AB \sin \alpha \sin \beta}{\sin (\alpha - \beta)}, \quad (178)$



HEIGHTS AND DISTANCES

$$\frac{AD}{AB} = \frac{\sin \alpha}{\sin (\alpha + \beta)}$$

2^c. Suppose the points
 A, B, D not collinear.

Let the angle of elevation at A be denoted by α , and the angles at A and B of the triangle ABD by α' , β respectively; then in the triangle ABD , angle ABD , we have

$$AD : AB :: \sin \beta : \sin D \text{ or } \sin (\alpha' + \beta)$$

$$\text{Hence } AD = \frac{AB \sin \beta}{\sin (\alpha' + \beta)}$$

$$\text{but } CD : AD :: \tan \alpha$$

$$\text{Hence } CD = \frac{AB \tan \alpha \sin \beta}{\sin (\alpha' + \beta)}$$

This enables us to find the length of a base line.

77. To Find the Distance between two Inaccessible Objects on a Horizontal Plane.

Let C, D be the inaccessible objects; AB the base line; then, measuring with a theodolite or other instrument the angles marked $\alpha, \alpha', \beta, \beta'$ in the diagram; in the triangle ABC we have

$$BC : AB :: \sin \alpha : \sin C \text{ or } \sin (\alpha + \beta')$$



therefore

$$BC = \frac{AB \sin \alpha}{\sin(\alpha + \beta')}.$$

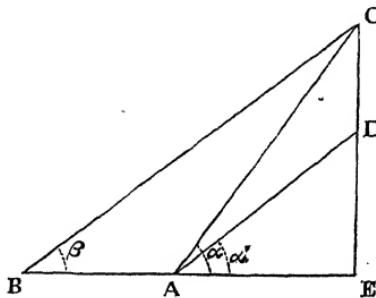
In like manner, from the triangle ABD we get

$$BD = \frac{AB \sin \alpha'}{\sin(\alpha' + \beta)};$$

These equations give us the sides BC , BD of the triangle BCD , and the contained angle is $(\beta - \beta')$. Hence CD can be found by Case III. of the solution of triangles.

78. To find the Height of an Inaccessible Object situated above a Horizontal plane, and its Height above the Plane.

Let CD be the object; AB the base line; α , β the angles of elevation of C from A and B respectively;



α' the elevation of D from A . Then, from equation (178), we have

$$AE = \frac{AB \sin \alpha \cos \beta}{\sin(\alpha - \beta)}.$$

Hence

$$DE = \frac{AB \sin \alpha \cos \beta \tan \alpha'}{\sin(\alpha - \beta)}.$$

Again, from equation (175), we have

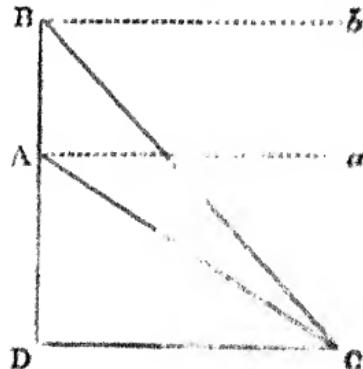
$$EC = \frac{AB \sin \alpha \cdot \sin \beta}{\sin(\alpha - \beta)}.$$

Hence

$$\begin{aligned} CD &= \frac{AB \sin \beta}{\sin(\alpha - \beta)} \{ \sin \alpha - \cos \alpha \tan \alpha' \} \\ &= \frac{AB \sin(\alpha - \alpha') \sin \beta}{\cos \alpha' \sin(\alpha - \beta)}. \end{aligned} \quad (181)$$

79. To find the Distance of an Object on a Horizontal Plane, from Observations made at two Points in the same Vertical, above the Plane.

Let A, B be the points of observation ; C the point observed, whose horizontal distance CD and vertical distance AD are required. Through AB draw the horizontal lines Aa, Bb . The angles aAc, bBc are called the angles of depression of C ; let these be denoted by δ, δ' . Now, it is evident that $BD = CD \tan \delta'$, $AD = CD \tan \delta$.



Hence $AB = CD (\tan \delta' - \tan \delta)$;

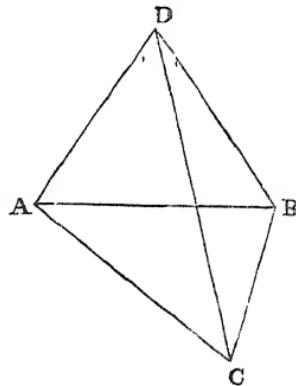
that is $AB = \frac{CD \sin(\delta' - \delta)}{\cos \delta' \cos \delta}$.

Hence $CD = \frac{AB \cdot \cos \delta \cos \delta'}{\sin(\delta' - \delta)}$; (182)

and therefore $AD = \frac{AB \sin \delta \cdot \cos \delta'}{\sin(\delta' - \delta)}$. (183)

80. *A, B, C are the angular points of a triangle, the lengths of whose sides are known; D is a point in the plane of the triangle, at which the sides AC, BC subtend given angles α, β ; it is required to find the distances AD, BD.*

Let the angles CAD, CBD be denoted by x, y , respectively; then, since the sum of the four angles of the quadrilateral $ACBD$ is four right angles, and the sum of the angles ADB, ACB is $(\alpha + \beta + C)$, the sum of the angles x, y is given. Now, denoting the sides of the given triangle by a, b, c , we have from the triangles ADC, BDC , respectively,



$$CD = \frac{b \sin x}{\sin \alpha}, \quad CD = \frac{a \sin y}{\sin \beta};$$

therefore
$$\frac{\sin x}{\sin y} = \frac{a \sin \alpha}{b \sin \beta}$$

Now, assume an auxiliary angle ϕ , such that

$$\tan \phi = \frac{a \sin \alpha}{b \sin \beta};$$

then ϕ can be found from the tables: thus we have

$$\frac{\sin x}{\sin y} = \frac{\tan \phi}{1}.$$

Hence
$$\frac{\sin x - \sin y}{\sin x + \sin y} = \frac{\tan \phi - 1}{\tan \phi + 1};$$

$$\text{therefore } \frac{\tan \frac{1}{2}(x-y)}{\tan \frac{1}{2}(x+y)} = \tan \left(\phi - \frac{\pi}{4} \right). \quad (184)$$

Hence $(x-y)$ is determined, since $(x+y)$ is known; therefore x and y can be found, and the question is solved.

EXERCISES.—XLII.

1. A river 300 feet wide runs at the foot of a tower, which subtend an angle of $22^{\circ} 30'$ at the edge of the remote bank; find the height of the tower.
2. At 375 feet from the foot of a temple the elevation is half what it is at 135 feet; find its height.
3. Find the height of a hill, the angle of elevation at its foot being 60° , and at a point 500 yards from the foot along a horizontal plane 30° .
4. If the length of a road, in which the ascent is 1 in 5, be 15 miles, what will be the length of a zigzag road which attains the same height, with an ascent of 1 in 12?
5. The shadow of a lamp-post 9 feet long is $3\sqrt{3}$ feet in sunlight; find the sun's altitude, and the height of a tower whose shadow is 120 feet.
6. A tower 51 feet high has a mark at a height of 25 feet from the ground; find at what distance from the foot the two parts subtend equal angles.
7. The sides of a horizontal triangle ABC are $AB = 300$, $BC = 399$, $AC = 108$; find the altitude of a tower CD if the angle ADC be $44^{\circ} 14' 48''$.
8. An object, whose height is AB , situated above a horizontal plane, subtends an angle a at a fixed point E in the plane; determine the angles of depression of E as seen from A and B .
9. The angles of a triangle are as $1 : 2 : 3$, and the perpendicular from the greatest angle on the opposite side is 30 yards; calculate the sides.

10. At two points A, B , an object DE , situated in the same vertical line CE , subtends the same angle α . If AC, BC be in the same right line, and equal to a and b respectively, prove

$$DE = (a + b) \tan \alpha.$$

11. From a station B at the foot of an inclined plane BC the angle of elevation of the summit A of a mountain is 60° , the inclination of BC is 30° , the angle BCA 135° , and the length of BC 1000 yards; find the height of A over B .

12. If a line AB subtends a right angle at the foot of a tower, and if the angles of elevation at A and B be 30° and 18° , respectively; prove

$$\text{height} = \frac{AB}{\sqrt{2} + 2\sqrt{5}}.$$

13. A spherical balloon, whose radius is r feet, subtends an angle α , when the angle of elevation of its centre is β ; prove

$$\text{height} = r \sin \beta \operatorname{cosec} \frac{1}{2}\alpha.$$

14. Find at what distance asunder two points, each 12 feet above the earth's surface, cease to be visible from each other, the earth's diameter being 7926 miles.

15. ABC is a right-angled triangle, of which C is the right angle; if the angular elevation of a steeple at A from B and C be 15° and 45° respectively, prove that

$$\tan B = \frac{1}{2} \{ 3^{\frac{1}{2}} - 3^{-\frac{1}{2}} \}.$$

16. If the angle of elevation of a cloud from a point h feet above a lake be α , and the angle of depression of its reflection in the lake β ; prove

$$\text{height} = \frac{h \sin \beta + \alpha}{\sin(\beta - \alpha)}.$$

17. From the top of a cliff h feet high the depressions of two ships at sea, in a line with the foot of the cliff, are δ , δ' , respectively; prove distance between the ships = $h (\cot \delta' - \cot \delta)$ feet.

18. Find the side of an equilateral triangle whose area costs as much in paving, at $8d.$ per square foot, as the palisading of the three sides at $7s.$ per foot.

19. The annual parallax of the star α Centauri is $.75''$, and the radius of the earth's orbit is 92,700,000 miles; find the distance of α Centauri.

20. A right-angled triangle rests on its hypotenuse, the length of which is 100 feet; one of the angles is 36° , and the inclination of the plane of the triangle to the horizon is 60° ; find the height of the vertex above the ground.

21. A station at A is due west of a railway train at B ; after travelling N.W. 6 miles, the bearing of A from the train is S.S.W.; required the distance AB .

22. From a point on a hillside the angle of elevation of an obelisk on its summit is α , and a foot nearer the top it is β ; if h be the height of the obelisk, prove inclination of hill to the horizon is

$$\cos^{-1} \left\{ \frac{\alpha \sin \alpha \sin \beta}{h \sin (\beta - \alpha)} \right\}.$$

23. A, B, C are telegraph posts at equal intervals by the side of a road; τ, τ' are the tangents of the angles which AB, BC subtend at a point P ; and T is the tangent of the angle which PB makes with the road; prove

$$\frac{2}{T} = \frac{1}{\tau} - \frac{1}{\tau'}.$$

24. A balloon is ascending uniformly and vertically; when it is one mile its angle of elevation is α ; and 15 minutes later it is β ; find the rate of ascent.

25. What is the dip of the horizon from the top of a mountain $2\frac{1}{2}$ miles high, the earth's radius being 3963 miles?

26. Resolve a right-angled triangle, being given the hypotenuse and the bisector of the right angle.

27. Resolve a right-angled triangle, being given the radius of the inscribed circle and the bisector of the right angle.

28. Resolve a triangle, being given the three angles and one of its altitudes.

29. Resolve a triangle, being given the angle A , the altitude h , and the corresponding median m .

30. Calculate the angles and the area of a trapezium, being given the parallel sides and the two diagonals.

31. Two inaccessible objects A , B are observed from two stations C and D , 1124 feet apart; the angle ACB is $62^\circ 12'$, $BCD 41^\circ 8'$, $ADC 34^\circ 51'$, and $ADB 60^\circ 49'$; find the distance AB .

32. If the sun's altitude be 47° , what angle must a stick make with the horizon that the length of its shadow may be a maximum?

33. A person travelling along a road takes the altitude of a tower, and also its angular distance, from an object in the road; the former is α° and the latter β° ; prove, if d be the nearest distance of the tower from the road, that its height is

$$\frac{d \sin \alpha}{\sqrt{\cos(\alpha + \beta) \cos(\alpha - \beta)}}.$$

34. A balloon seen from a certain station has an altitude of 50° , and bearing N.W.; what will be its bearing at a station south of the former, when the altitude of the balloon is 30° ?

35. Three stations A , B , C lie in a right line from south to north in the order given; if θ , ϕ denote the angles subtended by AB , BC , respectively, at a station D , prove that the bearing of B from D west of north is

$$\cot^{-1} \left\{ \frac{BC \cot \phi - AB \cot \theta}{AC} \right\}.$$

ANSWERS TO EXERCISES.

EXERCISES.—XII. PAGE 39.

- | | | | |
|----|--------------------|-----------------|----------------|
| 2. | (1) 2.1072100. | (2) 2.7092700. | (3) .5051500. |
| 3. | (1) 1.9084852. | (2) 3.3398491. | (3) .3856065. |
| 4. | (1) 2.5352940. | (2) 3.3803920. | (3) 1.2254900. |
| 5. | 1°. (1) 4.3167252. | (2) 2.6354839. | (3) 1.9912260. |
| | (4) 2.8363240. | (5) .2375439. | (6) 1.5263393. |
| | 2°. (1) .8116246. | (2) .7090810. | (3) .3598681. |
| | (4) .4322143. | (5) .2945469. | (6) 1.6153521. |
| | 3°. (1) 1.6505150. | (2) 1.6192803. | (3) 1.4644438. |
| | (4) 4.9736780. | (5) 17.1162465. | (6) 1.2631696. |

EXERCISES.—XIII. PAGE 43.

- | | | | |
|----|---------------------------|---------------------------|----------------|
| 1. | $A = 58^\circ 14' 54''$, | $B = 31^\circ 45' 6''$, | $c = 286.95$. |
| 2. | $A = 36^\circ 1' 10''$, | $B = 36^\circ 58' 50''$, | $c = 150.8$. |
| 3. | $A = 36^\circ 52' 11''$, | $B = 53^\circ 7' 49''$, | $c = 20$. |
| 4. | $A = 4^\circ 42' 0''$, | $B = 85^\circ 18' 0''$, | $c = 150$. |
| 5. | $A = 85^\circ 14' 0''$, | $B = 4^\circ 46' 0''$, | $c = 4650$. |

EXERCISES.—XIV. PAGE 44.

1. $A = 53^\circ 15' 7''$, $B = 36^\circ 44' 53''$, $b = 1312.7$.
2. $A = 63^\circ 6' 10''$, $B = 26^\circ 53' 50''$, $b = 171.46$.
3. $A = 31^\circ 24' 15''$, $B = 58^\circ 35' 45''$, $b = 7331.7$.
4. $A = 56^\circ 3' 0''$, $B = 33^\circ 57' 0''$, $b = 4832.4$.
5. $A = 27^\circ 10' 35''$, $B = 62^\circ 49' 25''$, $b = 1542.55$.

EXERCISES.—XV. PAGE 45.

1. $b = 391.79$, $c = 935.00$, $B = 24^\circ 46' 0''$.
2. $b = 4832.4$, $c = 8653.1$, $B = 33^\circ 57' 0''$.
3. $b = 474.07$, $c = 504.99$, $B = 69^\circ 50' 43''$.
4. $b = 675.30$, $c = 929.00$, $B = 46^\circ 37' 34''$.
5. $b = 1.4610$, $c = 1.7045$, $B = 58^\circ 59' 34''$.

EXERCISES.—XVI. PAGE 46.

1. $\alpha = 135$, $b = 154.27$, $B = 48^\circ 48' 43''$.
2. $\alpha = 792$, $b = 1542.55$, $B = 62^\circ 49' 25''$.
3. $\alpha = 507$, $b = 784.42$, $B = 57^\circ 7' 25''$.
4. $\alpha = 96.8412$, $b = 191.13$, $B = 63^\circ 7' 46''$.
5. $\alpha = 80.782$, $b = 193.96$, $B = 67^\circ 23' 22''$.

EXERCISES.—XXV. PAGE 64.

1. $2n\pi + \frac{\pi}{6}$, or $(2n+1)\pi - \frac{\pi}{6}$.
2. $n\pi \pm \frac{\pi}{4}$.
3. $n\pi \pm \alpha$.
4. $n\pi \pm \frac{\pi}{3}$.

EXERCISES.—XXVI. PAGE 65.

2. $\theta = \frac{2n\pi}{3}.$

3. $\theta = \frac{2n+1}{6}\pi, \text{ or } \theta = n\pi \pm \frac{\pi}{3}.$

4. $\theta = \frac{n\pi}{2}, \text{ or } \theta = 2n\pi \pm \frac{\pi}{3}.$

5. $\theta = n\pi, \text{ or } = n\pi \pm \frac{\pi}{6}.$

6. $\theta = \frac{n\pi}{4}, \text{ or } = \frac{2n\pi}{5} \pm \frac{\pi}{15}.$

7. $\theta = 2n\pi \pm \frac{\pi}{3} + \frac{\pi}{4}.$

8. $\theta = (2n+1)\frac{\pi}{2}, \text{ or } = \frac{n\pi}{5}.$

9. $\theta = \frac{(2n+1)\pi}{10}, \text{ or } = \frac{(2n+1)\pi}{8}, \text{ or } = \frac{(2n+1)\pi}{6}.$

10. $\theta = 2n\pi \pm \frac{\pi}{3}.$

11. $\theta = (4n-1)\frac{\pi}{8}, \text{ or } = (4n-1)\frac{\pi}{4}.$

12. $\theta = (2n+1)\frac{\pi}{4}, \text{ or } = \frac{2n\pi}{9} + \frac{\pi}{54}, \text{ or } = \frac{(2n+1)\pi}{9} - \frac{\pi}{54}.$

EXERCISES.—XXVII. PAGE 67.

1. $\theta = n\pi + \frac{\pi}{4}.$

2. $\theta = n\pi \pm \frac{\pi}{3}.$

3. $\theta = n\pi \pm \frac{\pi}{6}.$

4. $\theta = n\pi \pm \frac{\pi}{4}.$

5. $\theta = n\pi$, or $= 2n\pi + \frac{\pi}{3}$.

6. $\theta = n\pi \pm \frac{\pi}{3}$.

7. $\theta = n\pi \pm \frac{\pi}{3}$.

8. $\theta = (n + \frac{1}{2})\pi$, or $= \frac{n\pi}{4} \pm \frac{\pi}{24}$.

9. $\theta = n\pi \pm \frac{\pi}{4}$.

10. $\theta = 2n\pi \pm \frac{\pi}{3}$.

11. $\theta + \frac{\pi}{4} = n\pi \pm \frac{\pi}{3}$.

12. $\theta = n\pi - \frac{\pi}{4}$, or $-n\pi$.

13. $\theta = 2n\pi + \frac{\pi}{6}$, or $(2n + 1)\pi - \frac{\pi}{6}$.

14. $\theta = \frac{n\pi}{2} \pm \frac{\pi}{24}$.

15. $\theta = 2n\pi \pm \frac{\pi}{6}$.

16. $\theta = 2n\pi \pm \frac{\pi}{3}$.

EXERCISES.—XXVIII. PAGE 68.

11. $x = \pm n$.

12. $x^2 = \frac{2}{17}(5 - 2\sqrt{2})$.

13. $x = -1$ or $\frac{1}{6}$.

14. $x = \frac{a+b}{1-ab}$; $x = 0$, or $\pm \frac{1}{2}$.

EXERCISES.—XXIX. PAGE 72.

1-3. $x = 2n\pi + \frac{\pi}{2}$; $x = \tan^{-1} \left(\frac{c - \sin a}{\cos a - b} \right)$; $x = \tan^{-1} \left(\frac{m - \sin a}{n + \cos a} \right)$.

4. $x = (2n + 1) \frac{\pi}{2}$, or $\sin^{-1} \left(\pm \frac{\cos a}{\sqrt{2}} \right)$.

5. $x = \frac{\pi}{4}$.

6. $x = n\pi$, or $\tan^2 x = \frac{1 \pm \sqrt{17}}{8}$.

7. $x = n\pi + \frac{\pi}{4}$, or $= \frac{n\pi}{2}$.

8. $x = \frac{n\pi}{3}$.

EXERCISES.—XXXVI. PAGE 99.

1. $b = 542.850$, $c = 595.638$, $A = 7^\circ 21'$.

2. $b = 331.657$, $c = 392.473$, $C = 54^\circ 18'$.

3. $b = 567.688$, $c = 663.986$, $C = 55^\circ 20'$.

4. $b = 630.771$, $c = 929.480$, $A = 77^\circ 0'$.

5. $b = 5686.00$, $c = 5357.50$, $B = 56^\circ 56'$.

EXERCISES.—XXXVII. PAGE 102.

1. $B = 46^\circ 52' 10''$, $C = 111^\circ 53' 25''$, $c = 883.65$,
or $133^\circ 7' 50''$. or $25^\circ 37' 45''$. or 411.92 .

2. $B = 51^\circ 9' 6''$, $C = 87^\circ 37' 54''$, $c = 116.82$,
or $128^\circ 50' 54''$. or $9^\circ 56' 6''$. or 20.172 .

3. $B = 24^\circ 51' 54''$, $C = 133^\circ 47' 41''$, $c = 615.67$,
or $155^\circ 2' 6''$. or $3^\circ 43' 29''$. or 55.41 .

4. $A = 66^\circ 5' 10''$, $C = 45^\circ 44' 26''$, $c = 65.696$.

5. $A = 42^\circ 53' 34''$, $C = 77^\circ 5' 54''$, $c = 39.453$.

EXERCISES.—XXXVIII. PAGE 103.

1. $A = 85^\circ 11' 58''$, $B = 79^\circ 36' 40''$, $c = 61$.
2. $A = 32^\circ 28' 19''$, $B = 21^\circ 19' 27''$, $c = 7713.3$.
3. $A = 67^\circ 28' 51.5''$, $B = 57^\circ 19' 5.5''$, $c = 18.41$.
4. $A = 136^\circ 15' 48''$, $B = 31^\circ 9' 4''$, $c = 2.69$.
5. $A = 133^\circ 51' 34''$, $B = 11^\circ 59' 10''$, $c = 2479.2$.

EXERCISES.—XXXIX. PAGE 105.

1. $A = 35^\circ 18' 0''$, $B = 76^\circ 18' 52''$, $C = 68^\circ 23' 8''$.
2. $A = 23^\circ 32' 12''$, $B = 56^\circ 8' 42''$, $C = 100^\circ 19' 6''$.
3. $A = 7^\circ 37' 42''$, $B = 61^\circ 55' 38''$, $C = 110^\circ 26' 40''$.
4. $A = 42^\circ 30' 44''$, $B = 49^\circ 25' 49''$, $C = 88^\circ 3' 27''$.
5. $A = 33^\circ 15' 39''$, $B = 50^\circ 56' 0''$, $C = 95^\circ 48' 21''$.
6. $A = 37^\circ 22' 19''$, $B = 38^\circ 15' 41''$, $C = 104^\circ 22' 0''$.

EXERCISES.—XL. PAGE 105.

1. $A = 27^\circ 47' 45''$, $B = 32^\circ 12' 15''$, $C = 13$.
2. $A = 59^\circ 37' 18''$, $B = 21^\circ 28' 42''$, $C = 590.92$.
3. $B = 90^\circ 0' 0''$, $C = 72^\circ 0' 0''$, $c = 3\sqrt[3]{(5+2\sqrt{5})}$.
4. $B = 45^\circ 0' 0''$, $C = 120^\circ 0' 0''$, $c = 5\sqrt[5]{(6+5\sqrt{3})}$.
7. $20\sqrt{3}$ acres.
11. 6 , 21 , 5π , $\frac{144\pi}{5}$, 45π , 80π .

EXERCISES.—XLI. PAGE 113.

1. 124.26 feet.
2. 180 feet.
3. $250\sqrt{3}$ yards.
4. 4 miles.
5. 30° , $40\sqrt{3}$ feet.
6. $25\sqrt{51}$ feet.

7. If the sides of the triangle be denoted by a , b , c , the height of the tower by h , and the angle ADB by δ ; then

$$c^2 = a^2 + b^2 + 2h^2 - 2\sqrt{(a^2 + h^2)(b^2 + h^2)} \cos \delta.$$

9. $20\sqrt{3}$, 60, $40\sqrt{3}$.
11. $500(3 + \sqrt{3})$.
14. $6\sqrt{2}$ miles nearly.
18. $42\sqrt{3}$ feet.
19. 50988708000000 miles.
20. $25\sqrt{3} \cos 18^\circ$.
21. 6 miles.
24. $\frac{4 \sin(\beta - \alpha)}{\sin \alpha \cos \beta}$ miles per hour.
25. $2^\circ 2' 22''$.
31. 1459.4 feet.
32. 43° .
34. N.N.W. $\frac{1}{4}$ N.

	SINE.	SECANT.	TANGENT.	
r	COSINE.	COSECANT.	COTANGENT.	
0°	0.0000000	1.0000000	0.0000000	90°
1°	0.0174524	1.0001523	0.0174551	89°
2°	0.0348995	1.0006095	0.0349208	88°
3°	0.0523360	1.0013723	0.0524078	87°
4°	0.0697565	1.0024419	0.0699268	86°
5°	0.0871557	1.0038198	0.0874887	85°
6°	0.1045285	1.0055083	0.1051042	84°
7°	0.1218693	1.0075098	0.1227846	83°
8°	0.1391731	1.0093276	0.1405408	82°
9°	0.1564345	1.0124651	0.1583844	81°
10°	0.1736482	1.0154266	0.1763270	80°
11°	0.1908090	1.0187167	0.1943803	79°
12°	0.2079117	1.0223406	0.2122566	78°
13°	0.2249511	1.0263041	0.2308042	77°
14°	0.2419219	1.0306136	0.2493210	76°
15°	0.2588190	1.0352762	0.2679492	75°
16°	0.2756374	1.0402694	0.2867454	74°
17°	0.2923717	1.0456918	0.3057307	73°
18°	0.3090170	1.0514622	0.3249197	72°
19°	0.3255682	1.0576207	0.3443276	71°
20°	0.3420201	1.0641778	0.3639702	70°
21°	0.3583679	1.0711450	0.3838640	69°
22°	0.3746066	1.0785347	0.4040262	68°
23°	0.3907311	1.0863604	0.4244748	67°
24°	0.4067306	1.0940363	0.4452237	66°
25°	0.4226183	1.1033779	0.4663077	65°
26°	0.4383711	1.1126019	0.4877326	64°
27°	0.4539995	1.1223262	0.5095234	63°
28°	0.4694716	1.1325701	0.5317091	62°
29°	0.4848096	1.1433541	0.5543991	61°
30°	0.5000000	1.1547005	0.5773503	60°

NATURAL SINES, ETC.

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	SINE.	SECANT.	TANGENT.	
30°	0.5000000	1.1547005	0.5773503	60°
31°	0.5150381	1.1666334	0.6008606	59°
32°	0.5299193	1.1791784	0.6248694	58°
33°	0.5446390	1.1923633	0.6494076	57°
34°	0.5591929	1.2062179	0.6745085	56°
35°	0.5735764	1.2207746	0.7002075	55°
36°	0.5877853	1.2360680	0.7265425	54°
37°	0.6018150	1.2521357	0.7535541	53°
38°	0.6156615	1.2690182	0.7812856	52°
39°	0.6293204	1.2867596	0.8097840	51°
40°	0.6427876	1.3054073	0.8390996	50°
41°	0.6560590	1.3250130	0.8692867	49°
42°	0.6691306	1.3456327	0.9004040	48°
43°	0.6819984	1.3673275	0.9325151	47°
44°	0.6946584	1.3901636	0.9656888	46°
45°	0.7071068	1.4142136	1.0000000	45°
46°	0.7193398	1.4395565	1.0355303	44°
47°	0.7313537	1.4662792	1.0723687	43°
48°	0.7431448	1.4944765	1.1106125	42°
49°	0.7547096	1.5242531	1.1503444	41°
50°	0.7660444	1.5557238	1.1917536	40°
51°	0.7771460	1.5890157	1.2348972	39°
52°	0.7880108	1.6242692	1.2799416	38°
53°	0.7986355	1.6616401	1.3270448	37°
54°	0.8090170	1.7013016	1.3763819	36°
55°	0.8191520	1.7434468	1.4281480	35°
56°	0.8290376	1.7882916	1.4825610	34°
57°	0.8386706	1.8360785	1.5398650	33°
58°	0.8480481	1.8870799	1.6003345	32°
59°	0.8571673	1.9416040	1.6642795	31°
60°	0.8660254	2.0000000	1.7320508	30°
	COSINE.	COSECANT.	COTANGENT.	

	SINE.	SECANT.	TANGENT.	
	COSINE.	COSECANT.	COTANGENT.	
60°	0.8660254	2.0000000	1.7320508	30°
61°	0.8746197	2.0626653	1.8040478	29°
62°	0.8820476	2.1309545	1.8807205	28°
63°	0.8910065	2.2026893	1.9626105	27°
64°	0.8987040	2.2811720	2.0501031	26°
65°	0.9063078	2.3662016	2.1415069	25°
66°	0.9135455	2.4585933	2.2460368	24°
67°	0.9205019	2.5503047	2.3555524	23°
68°	0.9271839	2.6694672	2.4750879	22°
69°	0.9335804	2.7904281	2.6050891	21°
70°	0.9396926	2.9238044	2.7474774	20°
71°	0.9455186	3.0715535	2.9042109	19°
72°	0.9510505	3.2360680	3.0776835	18°
73°	0.9563048	3.4203036	3.2704526	17°
74°	0.9612017	3.6279553	3.4874144	16°
75°	0.9659258	3.8637033	3.7420592	15°
76°	0.9702057	4.1335655	4.0107809	14°
77°	0.9743701	4.4454115	4.3314759	13°
78°	0.9781476	4.8097313	4.7046391	12°
79°	0.9816272	5.2403431	5.1445549	11°
80°	0.9841079	5.7587705	5.6712818	10°
81°	0.9876883	6.3924532	6.3137515	9°
82°	0.9902681	7.1852965	7.1153697	8°
83°	0.9925462	8.2055090	8.1443494	7°
84°	0.9945219	9.5667722	9.5143645	6°
85°	0.9961947	11.473713	11.430052	5°
86°	0.9975641	14.335557	14.300666	4°
87°	0.9981295	19.107323	19.081437	3°
88°	0.9993908	28.653708	28.636253	2°
89°	0.9998477	57.298688	57.289962	1°
90°	1.0000000	Infinite.	Infinite.	0°

APPENDIX.

EXAMINATION PAPERS.

I.—INTERMEDIATE, 1879.

Examiner.—PROF. B. WILLIAMSON, F.T.C.D., F.R.S.

1. Show how to construct an angle whose sine is given.
Construct an angle whose sine is $\frac{\sqrt{2}}{3}$.
2. Explain what is meant by the circular measure of an angle, and find the circular measure of $2^o 30'$.
3. If $\sin A = \frac{3}{5}$, and $\sin B = \frac{5}{13}$, find the value of $\cos(A + B)$.
4. Prove by the aid of a construction the equation

$$\cos(A + B) = \cos A \cos B - \sin A \sin B.$$
- Find what this formula becomes (1) if $A = B$; (2) if $A = 90^\circ$.
5. Express $(\sin A - \sin^3 A)^2 + (\cos A - \cos^3 A)^2$ in its simplest form.
6. Find the simplest values of A and B which satisfy the equations $\sin(3A + 4B) = \frac{1}{2}$, $\sin(7B + 5A) = 1$.
7. In a right-angled triangle, being given one side and the opposite angle, show how to find the remaining parts.
8. In a plane triangle, being given two sides, and the angle opposite to one of them; show how the remaining parts are found; point out in what case the solution is ambiguous, and when impossible.
9. Being given two sides and the contained angle, show how to determine the remaining parts.
10. In a plane triangle, prove $\tan \frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{s(s - a)}}$, and write down the corresponding formula in logarithms.
- *11. If $A + B + C = 180^\circ$, prove

$$\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C = 1.$$
- *12. Prove that $\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + \sin^{-1}\left(\frac{1}{\sqrt{10}}\right) = \frac{\pi}{4}$.
- *13. Solve the equation $\cos^2 \theta = \sin \theta$, and show that one of its roots gives an impossible solution.

II.—INTERMEDIATE, 1881.

Examiner.—PROF. BURNSIDE, F.T.C.D.

1. Show geometrically how $\sin 2A$ and $\cos 2A$ can be expressed in terms of $\sin A$ and $\cos A$.

2. Express $\sin A$ and $\cos A$ in terms of $\sin 2A$.

3. Express the length of the perpendicular drawn from a vertex of a triangle to the opposite side in terms of the sides, proving any trigonometrical formula made use of in deducing the result.

4. Find the value of $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8}$.

5. Find the area of the triangle whose sides are 25 and 30 feet, and the included angle 54° .

6. Solve the trigonometrical equation $3 \sin x = 2 \cos^2 x$.

7. Express the radius of the inscribed circle of a triangle in terms of its sides.

8. Determine the relation which exists between the sides of a triangle in each of the following cases :—

$$(1) \sin^2 A = \sin^2 B + \sin^2 C. \quad (2) \sin A = 2 \cos B \sin C.$$

9. In any triangle, prove

$$\cos A + \cos B = 2 \sin^2 \frac{1}{2} C \cdot \frac{a+b}{c}.$$

10. In any plane triangle, prove the formula

$$\frac{\tan B}{\tan C} = \frac{a^2 + b^2 - c^2}{a^2 + c^2 - b^2}.$$

11. Find in its simplest form the area of the triangle whose sides are—

$$\sqrt{\beta^2 + \gamma^2}, \quad \sqrt{\gamma^2 + \alpha^2}, \quad \sqrt{\alpha^2 + \beta^2}.$$

12. Given $\tan(A + a) = p$, and $\tan(A - a) = q$; find $\sin 2A$ and $\cos 2A$ in terms of p and q .

III.—INTERMEDIATE, 1882.

Examiner.—PROF. CROFTON, F.R.S.

1. State the formula for $\sin(A + B)$, and deduce from it that for $\cos(A + B)$.
 2. Express $\cot A - \tan A$ as a monomial.
 3. Prove that the value of $\sin^2 \alpha + \sin^2 \beta - 2 \sin \alpha \sin \beta \sin(\alpha + \beta)$ is unchanged, if $\frac{\pi}{4} - \alpha, \frac{\pi}{4} - \beta$ are put for α and β , respectively.
 4. Find the area of the triangle whose sides are 12, 13, 5.
 5. Given the sides a, b , and the vertical angle C , of a triangle; express the length of a line bisecting that angle, and terminated by the base.
 6. Given a, b, C , find the side c ; also the length of the bisector of c drawn from C .
 7. If in a triangle $B = 2A$; find what relation exists between the three sides.
 8. A line passes through the centre of an equilateral triangle, and is terminated by the sides; if x, y be the segments into which it is divided at the centre, and a a side of the triangle, prove
- $$\frac{1}{x^2} - \frac{1}{xy} + \frac{1}{y^2} = \frac{9}{a^2}.$$
9. Explain the ambiguous case in the solution of triangles. Given $A = 60^\circ, a = 5, b = 6$; solve the triangle.
 10. Two sides of a triangle are 700 and 500 yards, and the contained angle $72^\circ 40'$; find the remaining angles; given

$$\log 9 = .9542425;$$

$$\log 2 = .3010300;$$

$$\text{Log tan } 53^\circ 40' = 10.1334356; \quad \text{Log tan } 12^\circ 46' = 9.3552267.$$

IV.—INTERMEDIATE, 1883.

Examiner.—PROF. LARMOR, D.Sc.

1. Define the sine, cosine, and tangent of an angle, and show how they differ from those of the same angle plus two right angles.
2. Prove $\sin(A + B) = \sin A \cos B + \cos A \sin B$. Find the results of writing—
 - (1) $90^\circ - A$ for A .
 - (2) $90^\circ - A$ for A , in this formula.
3. Express in decimals the area of a regular pentagon whose side is one foot.
4. A tower subtends an angle of 45° at a certain point in a level plain, and an angle of 30° at another point, which is 100 feet more distant from it; find its height.
5. Prove $\tan(\alpha + \beta) + \tan(\alpha - \beta) = \frac{2 \sin 2\alpha}{\cos 2\alpha + \cos 2\beta}$.
 Simplify $\frac{\cos \frac{1}{2}(\alpha + \beta) - \cos \frac{1}{2}(\alpha - \beta)}{\cos \frac{1}{2}(\alpha + \beta) + \cos \frac{1}{2}(\alpha - \beta)}$.
6. Prove that the length of the circumference of a circle lies between those of an inscribed and circumscribed polygon; hence, taking regular polygons—(1) of 6 sides, and (2) of 12 sides; find the limits between which π must lie.
7. Given $\tan A = 100$; find $\tan \frac{A}{2}$ to four decimal places; account for the double result.
8. Express the sine of an angle in terms of the sides. If the sides are 7, 8, 9, find the sine of the greatest angle.
9. Given the four sides of a quadrilateral, and one angle; show fully how to find the other angles.
10. The two sides of a triangle are 45 and 35 feet, and the contained angle is 120° ; find the length of the base, and the length of the straight line bisecting the vertical angle and terminated by the base, each to two places of decimals.

V.—WOOLWICH—Preliminary, June, 1882.

1. Prove that the angle subtended at the centre of a circle by an arc equal to the radius is invariable.
2. Define the secant of an angle, and apply your definition to angles in the third quadrant.
3. Obtain a formula embracing all angles having a given tangent.

Determine all the values of θ which satisfy the equation

$$1 + \sqrt{3} \cdot \tan^2 \theta = (1 + \sqrt{3}) \tan \theta.$$

4. Find an expression for $\tan 3A$ in terms of $\tan A$, and show that $\tan 3A \cdot \tan 2A \cdot \tan A = \tan 3A - \tan 2A - \tan A$.

5. Prove $\sin^2 3\theta = \sin 18 \sin 54$; and show that in any circle, the chord of an arc of 108° is equal to the sum of the chords of arcs of 36° and 60° .

6. Demonstrate the identities

$$\frac{(\operatorname{cosec} A + \sec A)^2}{(\operatorname{cosec}^2 A + \sec^2 A)} = 1 + \sec 2A; \quad 4(\cot^2 3 + \operatorname{cosec}^2 \frac{1}{5}\pi) = \pi.$$

7. What are the advantages gained by the use of logarithms, calculated to the base 10?

If $\log_{10} 2 = .3010300$, find $\log_{10} 5$, $\log_{10} 1\frac{1}{2}$, and $\log_{10} 4\sqrt{.005}$.

8. In any triangle, prove that

$$\frac{2bc \cos A}{a^2 + b^2 + c^2} = \frac{1 + \cos(A+B+C)}{1 + \cos(A+B-C)} = \frac{a^2 + b^2}{a^2 + c^2}.$$

9. If r_1 be the radius of the escribed circle which touches the side a of a triangle externally; prove

$$r_1 \cos \frac{A}{2} = a \cos \frac{B}{2} \cos \frac{C}{2}.$$

10. If a be the side of a regular polygon of n sides, R, r the radii of its circumscribed and inscribed circles; prove

$$R + r = \frac{a}{2} \cot \frac{\pi}{2n}.$$

11. Two sides of a triangle are respectively 250 and 200 yards long, and contain an angle of $54^\circ 36' 24''$; find the other angles, having given

$\operatorname{Log} \cot 27^\circ 18' = 10.2872338$; diff. for $t = .3100$;

$\operatorname{Log} \tan 12^\circ 8' 50'' = 9.3329292$; $\operatorname{Log} 3 = .4771213$.

VI.—WOOLWICH—Preliminary, December, 1882.

1. Find correct, to three places of decimals, the radius of a circle, in which an arc, 15 inches long, subtends at the centre an angle containing $71^\circ 30' 3.6''$.

2. Given $\cos A = .28$, determine the value of $\tan \frac{1}{2}A$, and explain the reason of the ambiguity which presents itself in your result.

3. Prove that

$$(1) \tan \theta + \cot \theta = 2\sec^2 \theta + \operatorname{cosec}^2 \theta;$$

$$(2) \sec \theta - \tan \theta = \tan \left(\frac{\pi}{4} - \frac{\theta}{2} \right);$$

$$(3) \cos 20^\circ + \cos 100^\circ + \cos 130^\circ = 0;$$

$$(4) \cos^2 \frac{\alpha}{2} + \tan^2 \frac{\alpha}{2} = \sin^2 \frac{\alpha}{2}.$$

4. State and prove the rules by means of which you can determine, by inspection, the integral part of the logarithm of any number.

Given $\log 4900 = .6954817$; $\log 49604 = .6954904$; find the logarithms of 49010 , 499100 , 49990125 .

5. If $c = \sqrt{2}$; $A = 417^\circ$; $B = 45^\circ$, find all the other parts of the triangle.

6. Find the greatest angle of the triangle whose sides are 50, 60, 70, respectively, having given

$$\log b = .7781513; \log \cos 39^\circ 14' = 0.88909644; \text{diff. } r' = 1032.$$

7. Express the area of a triangle in terms of one side and its adjacent angles. Two angles of a triangular field are $22\frac{1}{2}^\circ$ and 45° respectively, and the length of the side opposite to the former is a furlong; prove that the field contains exactly two acres and a half.

8. If d_1, d_2, d_3 be the diameters of the three inscribed circles of a triangle; prove that

$$d_1 d_2 + d_2 d_3 + d_3 d_1 = (a + b + c)^2.$$

VII.—JUNIOR EXHIBITIONS (T. C. D.), 1882.

Examiner.—W. S. McCAY, F.T.C.D.

1. Construct an angle whose secant shall be $\sqrt{3}$.
2. A man is observed to subtend an angle of $8' 30''$; on approaching 100 yards nearer to him, he subtends $10'$; find his height and distance.
3. Prove that the distance of the sea horizon in miles is $\sqrt{1.5}h$, when h is the height of the observer's eye above the sea level.
4. Prove that the area of a circle is πr^2 .
5. Solve the equation

$$\sin x + \cos x = 2\sqrt{2} \sin(x + 45^\circ)$$

6. Prove the relation in a triangle

$$a \sin(B+C) + b \sin(C+A) + c \sin(A+B) = 0$$

7. The sides of a triangle are 2 , $\sqrt{6}$, $1 + \sqrt{3}$; find the angles.
8. The sides of a triangle are 3 , 5 , 6 ; find the radii of its inscribed and circumscribed circles.
9. Prove

$$4 \tan^{-1} \frac{1}{5} = \tan^{-1} \frac{1}{239} = \frac{\pi}{4}$$

10. Two sides a , b of a triangle are to each other as $9:7$, and the included angle is $64^\circ 12'$; find $\log \tan \frac{1}{2}(A+B)$, being given

$$\log 2 = 0.3010300; \quad \log \tan 57^\circ 54' = 10.2025255.$$

VIII.—JUNIOR EXHIBITIONS (T. C. D.), 1883.

Examiner.—REV. R. TOWNSEND, F.T.C.D.

1. Reduce to circular measure $11^{\circ} 27' 33''$; and to degrees, minutes, and seconds, the angle whose circular measure is $\frac{1}{5}$.
2. Calculate in feet the distance at which a sphere, of a yard radius, would subtend an angle of half a minute to the eye of an observer.
3. Given that $p = r \sin \theta$; $q = r \cos \theta$; find, in terms of r and θ , the roots of the equation

$$x^2 + 2px - q^2 = 0.$$

4. Given that $p = r \sec \theta$, $q = r \tan \theta$; find, in terms of r and θ , the roots of the equation

$$x^2 - 2px + q^2 = 0.$$

5. Assuming the general formulae for $\cos(A \pm B)$ and for $\sin(A \pm B)$, in terms of the sines and cosines of A and B ; prove from them that

$$\cos 3A = 4 \cos^3 A - 3 \cos A, \text{ and } \sin 3A = 3 \sin A - 4 \sin^3 A.$$

6. Prove, by actual involution, that

$$(4 \cos^3 A - 3 \cos A)^2 + (3 \sin A - 4 \sin^3 A)^2 = 1.$$

7. If d be the diameter of the circumcircle of the triangle ABC ; prove

$$a = d \sin A, \quad b = d \sin B, \quad c = d \sin C.$$

8. The three sides of a triangle are 13, 14, 15. Calculate exactly the radii of its circumscribed and inscribed circles.

9. Express the altitude h of a triangle in terms—(1) of the base c , and the two base angles A , B ; (2) of the two sides a and b , and the vertical angle C .

10. The head of the Nelson Column being supposed to have the altitudes α and β , at the distances a and b from the foot of the O'Connell Monument, in the line joining the feet of the monuments; required the height of the Nelson Column?

IX.—JUNIOR EXHIBITIONS (T. C. D.), 1884.

Examiner.—PROF. B. WILLIAMSON, F.T.C.D.

1. If $\tan A = 2 - \sqrt{3}$; find the value of $\tan 3A$.
2. Find the simplest form of the expression

$$\frac{\sin \theta - \sin 3\theta + \sin 5\theta}{\cos \theta - \cos 3\theta + \cos 5\theta}.$$

3. Find the value of x which satisfies the equation

$$\tan^{-1} \frac{x}{a} + \tan^{-1} \frac{x}{b} + \tan^{-1} \frac{x}{c} = \frac{\pi}{2}.$$

- *4. Find the real and the imaginary parts of the fraction

$$\frac{\cos \alpha + \sqrt{-1} \sin \alpha}{\cos \beta + \sqrt{-1} \sin \beta}.$$

5. In a plane triangle, prove the formula

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}},$$

and write down the corresponding logarithmic equation.

- *6. Explain the method of calculating the cosines of small angles, and find the value of $\cos 1^\circ$ to five decimal places.

7. Being given two sides of a triangle and its contained angle; show how to find the remaining sides and angles.

8. In a triangle, being given

$$a = 62, \quad b = 123, \quad c = 125;$$

find the numerical value of $\cos \frac{B}{2}$.

9. Prove, in any manner, that the trisection of an angle is reducible to the solution of a cubic equation.

10. Find the area of a quadrilateral inscribed in a circle in terms of the sides.

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